The Proportionality Principle of Simplicial Volume

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Hiermit versichere ich, daß ich diese Arbeit selbständig verfaßt und nur die angegebenen Quellen und Hilfsmittel benutzt habe.

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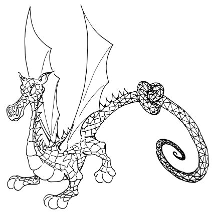

Introduction

Manifolds, the basic objects of geometry, form the playground for both topological and smooth structures. Many challenges in modern mathematics are concerned with the nature of this interaction between algebraic topology, differential topology and differential geometry. For example all incarnations of the index theorem, the Gauß-Bonnet formula, and the Mostow rigidity theorem fall into this category.

Theorem (Mostow Rigidity Theorem). *Each homotopy equivalence between oriented closed connected hyperbolic manifolds of dimension at least 3 is homotopic to an isometry.*

This theorem was first proved by Mostow [Mostow]. But there is also a proof by Gromov [Munkholm], for which he designed the simplicial volume (therefore also known as "Gromov norm").

The simplicial volume is a homotopy invariant of oriented closed connected manifolds defined in terms of the singular chain complex with real coefficients. This invariant consists of a nonnegative real number measuring the efficiency of representing the fundamental class using singular simplices: If M is an oriented closed connected manifold, then the *simplicial volume* of M, denoted by $\|M\|$, is the infimum of all ℓ^1-norms of singular cycles with real coefficients representing the fundamental class. Since the fundamental class can be viewed as a generalised triangulation of the manifold, the simplicial volume can also be interpreted as a measure for the complexity of a manifold.

But the simplicial volume also has another important interpretation: it is a homotopy invariant approximation of the Riemannian volume. For example, Gromov's proof of the Mostow rigidity theorem is based on the following proportionality principle for hyperbolic manifolds [Gromov_2, Section 2.2], [Thurston, Theorem 6.2]:

Theorem. *For $n \in \mathbf{N}$ define $v_n \in \mathbf{R}_{>0}$ as the maximal volume of an ideal n-simplex in n-dimensional hyperbolic space. Then for each hyperbolic oriented closed connected manifold M of dimension n,*

$$\frac{\|M\|}{\operatorname{vol}(M)} = \frac{1}{v_n}.$$

In particular, the simplicial volume does not always vanish. However, in general the simplicial volume is rather hard to compute. In the most cases where the simplicial volume is known, it is zero. This, for example, is the case if the fundamental group of the manifold is Abelian or, more generally, amenable.

One of the most complete (but hardly digestible) references about the simplicial volume is Gromov's pioneering paper "Volume and Bounded Cohomology" [Gromov_2]. More recent introductions are given in the textbooks on hyperbolic geometry by Ratcliffe and Benedetti, Petronio [Ratcliffe, §11.5], [Benedetti, Petronio, Section C.3].

Applications of the simplicial volume also include knot theory [Murakami], the existence of certain S^1-operations [Yano], and the investigation of other volumes, such as the Riemannian or the minimal volume [Gromov_2]. Furthermore, Gromov conjectured the following relation between the simplicial volume and L^2-invariants [Gromov_3, Section 8A] – thereby connecting two distant fields of algebraic topology:

Conjecture. *Let M be an aspherical closed orientable manifold whose simplicial volume vanishes. Then for all $p \in \mathbf{N}$,*

$$b_p^{(2)}\big(\widetilde{M}; \mathcal{N}(\pi_1(M))\big) = 0.$$

A rather complete account of L^2-invariants is given in the book *L^2-Invariants: Theory and Applications to Geometry and K-theory* [Lück_2]. The above conjecture is justified by the observation that L^2-invariants and the simplicial volume show a similar behaviour. We will discuss this conjecture briefly in Section 1.4. Among the most striking similarities is the proportionality principle:

Theorem (Proportionality Principle of L^2-Invariants). *Let U be a simply connected Riemannian manifold. Then for each $p \in \mathbf{N}$ there are constants $B_p \in \mathbf{R}_{>0}$ and a constant $T \in \mathbf{R}_{>0}$ satisfying: for any discrete group G acting freely and cocompactly on U by isometries and all $p \in \mathbf{N}$,*

$$b_p^{(2)}\big(U;\mathcal{N}(G)\big) = B_p \cdot \mathrm{vol}(G \setminus U).$$

If U with this G-action is of determinant class, then

$$\varrho^{(2)}\big(U;\mathcal{N}(G)\big) = T \cdot \mathrm{vol}(G \setminus U).$$

In particular, if M and N are oriented closed connected Riemannian manifolds with isometrically isomorphic universal Riemannian coverings, then for all $p \in \mathbf{N}$,

$$\frac{b_p^{(2)}\big(\widetilde{M};\mathcal{N}(\pi_1(M))\big)}{\mathrm{vol}(M)} = \frac{b_p^{(2)}\big(\widetilde{N};\mathcal{N}(\pi_1(N))\big)}{\mathrm{vol}(N)}$$

and

$$\frac{\varrho^{(2)}\big(\widetilde{M};\mathcal{N}(\pi_1(M))\big)}{\mathrm{vol}(M)} = \frac{\varrho^{(2)}\big(\widetilde{N};\mathcal{N}(\pi_1(N))\big)}{\mathrm{vol}(N)}.$$

Theorem (Proportionality Principle of Simplicial Volume). *Let M and N be oriented closed connected Riemannian manifolds with isometrically isomorphic universal Riemannian coverings. Then*

$$\frac{\|M\|}{\mathrm{vol}(M)} = \frac{\|N\|}{\mathrm{vol}(N)}.$$

These proportionality principles are also examples for an interesting link between topological and differential structures. A similar proportionality principle is known for Chern numbers [Hirzebruch, Satz 3].

A proof in the L^2-case is given in Lück's book [Lück$_2$, Theorem 3.183]. For the simplicial volume, sketches of (different) proofs were given by Gromov [Gromov$_2$, Section 2.3] and Thurston [Thurston, page 6.9]. However, there is no complete proof of the proportionality principle of simplicial volume in the literature. It is the aim of this diploma thesis to provide such a proof, based on Thurston's approach.

This text is organised as follows:

In the first chapter the simplicial volume is introduced and some of its properties are collected.

Chapter 2 is devoted to the study of bounded cohomology, a quite peculiar variant of singular cohomology. Its application as a basic tool for the analysis of the simplicial volume was discovered by Gromov [Gromov_2]. However, we will mostly refer to the more elegant approach due to Ivanov, based on classical homological algebra [Ivanov_1], [Monod].

Thurston's proof of the proportionality principle relies on the computation of the simplicial volume by means of a new homology theory called measure homology [Thurston, pages 6.6–6.9]. Instead of linear combinations of singular simplices, measures on the set of (smooth) singular simplices are considered as chains, thereby permitting averaging constructions such as Thurston's smearing. But Thurston's exposition lacks a proof of the fact that measure homology and singular homology give rise to the same simplicial volume. We will define measure homology in Chapter 3 and list its basic algebraic properties as they can be found in the papers of Zastrow and Hansen [Zastrow], [Hansen], in a slightly different setting.

Chapter 4 constitutes the central contribution of this diploma thesis. It will be shown that measure homology indeed can be used to compute the simplicial volume, i.e., that singular homology with real coefficients is *isometrically* isomorphic to measure homology. The proof is based on techniques from bounded cohomology, as explained in Chapter 2. It will also be discussed why Bowen's argument [Bowen] is not correct (Subsection 4.3.4).

The last chapter finally contains the proof of the proportionality principle – using Thurston's smearing technique. This forces us to study isometry groups of Riemannian manifolds. En passant, it will be proved that the compact open topology and the $\mathcal{C}^1$-topology (the smooth counterpart of the compact open topology) on the isometry groups coincide. Unfortunately, I was not able to show this without making use of the theory of standard Borel spaces. The chapter closes with a proof of the fact that the simplicial volume of a flat orientable closed connected manifold vanishes. The proof is based on the proportionality principle, instead of referring to Gromov's sophisticated estimate of the simplicial volume via the minimal volume [Gromov_2, page 220].

I tried to keep the prerequisites for the understanding of this diploma thesis as small as possible. So only a basic familiarity with singular (co)homology, covering theory, measure theory, and Riemannian geometry is required. A variety of textbooks is available providing such a background: e.g., [Lee_1], [Lee_2], [Massey], [Bredon], [Elstrodt]. But the detailed exposition of the subject comes at a price – the length of the text, for which I apologise.

The symbols $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ stand for the set of integers, the set of nonnegative integers, the set of rational numbers, the set of real numbers, and the set of complex numbers respectively. The one point space is written as $\bullet$. For simplicity, we always assume that manifolds have dimension at least 1. The (co)homology class represented by an element x is denoted by $[x]$. All sums "$\sum$" are implicitly finite, unless stated otherwise. All notation is collected in the table of notation (between bibliography and index).

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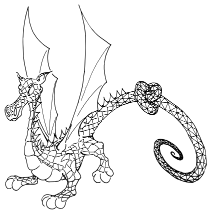


Contents

1 Simplicial Volume

The simplicial volume of an oriented closed connected manifold is a homotopy invariant – defined in terms of the singular chain complex – which measures the efficiency of representing the fundamental class by singular chains (with real coefficients). Its first appearance is in Gromov's famous proof of the Mostow rigidity theorem [Munkholm].

Despite of being a topological invariant the simplicial volume contains interesting information about the possible differential structures on a manifold – in form of the Riemannian volume. Examples for such relations are Gromov's estimate of the minimal volume and the proportionality principle, which is the topic of this thesis.

Moreover, there seems to be a deep connection to L^2-invariants [Gromov$_3$, Section 8A]:

Conjecture. *Let M be an aspherical closed orientable manifold whose simplicial volume vanishes. Then for all $p \in \mathbf{N}$,*

$$b_p^{(2)}\big(\widetilde{M}; \mathcal{N}(\pi_1(M))\big) = 0.$$

In this chapter the simplicial volume is introduced (Section 1.2) and some of its properties are collected (Section 1.3). We then give a short survey about indications supporting the above conjecture in Section 1.4. In the last section we briefly discuss some generalisations of the simplicial volume.

1.1 ■ Singular (co)homology refreshed

Before defining the simplicial volume, we recall the basic definitions of singular (co)homology and introduce the corresponding notation which will be used throughout this thesis. The key objects of singular (co)homology are simplices:

Definition (1.1). For $k \in \mathbf{N}$ the set

$$\Delta^k := \left\{ (x_0, \dots, x_k) \in \mathbf{R}^{k+1}_{\geq 0} \,\middle|\, \sum_{j=0}^{k} x_j = 1 \right\}$$

is called the k**-dimensional standard simplex**. For $j \in \{0, \dots, k\}$ we let

$$\begin{aligned} \partial_j \colon \Delta^{k-1} &\longrightarrow \Delta^k \\ (x_0, \dots, x_{k-1}) &\longmapsto (x_0, \dots, x_{j-1}, 0, x_j, \dots, x_{k-1}) \end{aligned}$$

be the inclusion of the j**-th face of** Δ^k. ◇

To be precise, the face mappings ∂_j also depend on k, but we omit k in the notation since it can easily be inferred from the context.

The main idea of the singular theory is to examine all continuous maps from the standard simplices to a topological space to detect the "holes" of this space, i.e., to measure which "cycles" are no "boundaries".

Definition (1.2). Let X be a topological space and $k \in \mathbf{N}$.

- The set $S_k(X)$ of **singular** k**-simplices** is the set of all continuous functions $\Delta^k \longrightarrow X$. If $\sigma \in S_k(X)$ and $j \in \{0, \dots, k\}$, then $\sigma \circ \partial_j$ is called the j**-th face of** σ.

- Let R be a commutative ring with unit. Then the set $C_k(X, R)$ of **singular** k**-chains of** X **with coefficients in** R is the free R-module generated by $S_k(X)$.

- The **boundary operator** $\partial\colon C_{k+1}(X,R) \longrightarrow C_k(X,R)$ is the R-module homomorphism given on the generators $\sigma \in S_{k+1}(X)$ by
$$\partial(\sigma) := \sum_{j=0}^{k+1} (-1)^j \cdot \sigma \circ \partial_j.$$
The elements of $\operatorname{im}\partial$ are called – as geometry suggests – **boundaries**, and the elements of $\ker\partial$ are called **cycles**.

- We define the set $C^k(X,R)$ of **singular k-cochains of X with coefficients in R** as the algebraic dual of $C_k(X,R)$, i.e.,
$$C^k(X,R) := \operatorname{Hom}_{\mathbf{Z}}\big(C_k(X,\mathbf{Z}),R\big) = \operatorname{Hom}_R\big(C_k(X,R),R\big)$$

- The boundary operator $\partial\colon C_{k+1}(X,R) \longrightarrow C_k(X,R)$ induces the **coboundary operator** $\delta\colon C^k(X,R) \longrightarrow C^{k+1}(X,R)$ as follows:
$$\begin{aligned}\delta\colon C^k(X,R) &\longrightarrow C^{k+1}(X,R)\\ f &\longmapsto \big(c \mapsto (-1)^{k+1}\cdot f(\partial c)\big).\end{aligned}$$
The elements of $\operatorname{im}\delta$ are called **coboundaries** and the elements of $\ker\delta$ are called **cocycles**.

In the special case $R = \mathbf{Z}$, we use the abbreviations $C_k(X) := C_k(X,\mathbf{Z})$ and $C^k(X) := C^k(X,\mathbf{Z})$. ◇

A lengthy calculation shows that in the situation of the above definition the equality $\partial \circ \partial = 0$ holds (i.e., the singular chains form a chain complex). Therefore $\delta \circ \delta = 0$, too. Hence the following definition makes sense:

Definition (1.3). Let X be a topological space, $k \in \mathbf{N}$, and let R be a commutative ring with unit.

- The R-module
$$H_k(X,R) := \frac{\ker\big(\partial\colon C_k(X,R) \to C_{k-1}(X,R)\big)}{\operatorname{im}\big(\partial\colon C_{k+1}(X,R) \to C_k(X,R)\big)}$$
is called **k-th singular homology of X with coefficients in R**.

- The R-module
$$H^k(X,R) := \frac{\ker\big(\delta\colon C^k(X,R) \to C^{k+1}(X,R)\big)}{\operatorname{im}\big(\delta\colon C^{k-1}(X,R) \to C^k(X,R)\big)}$$
is called **k-th singular cohomology of X with coefficients in R**.

For the sake of brevity we write

$$H_k(X) := H_k(X, \mathbf{Z}),$$
$$H^k(X) := H^k(X, \mathbf{Z}). \qquad \diamond$$

For convenience, we introduce the following convention:

Definition (1.4). Let M be an R-module, $B \subset M$, and $(a_b)_{b \in B} \subset R$. When we write $\sum_{b \in B} a_b \cdot b$, we implicitly assume that all but a finite number of the coefficients $(a_b)_{b \in B}$ are 0 unless stated otherwise. Additionally, whenever we write something like

"Let $\sum_{\sigma \in S_k(X)} a_\sigma \cdot \sigma \in C_k(X, R)$."

it is understood that all a_σ are elements of R. $\diamond$

Definition (1.5). Let $f\colon X \longrightarrow Y$ be a continuous map and let R be a commutative ring with unit. The map f induces chain maps

$$\begin{aligned} C_k(f, R)\colon C_k(X, R) &\longmapsto C_k(Y, R) \\ \sum_{\sigma \in S_k(X)} a_\sigma \cdot \sigma &\longmapsto \sum_{\sigma \in S_k(X)} a_\sigma \cdot f \circ \sigma \end{aligned}$$

and

$$\begin{aligned} C^k(f, R)\colon C^k(Y, R) &\longmapsto C^k(X, R) \\ g &\longmapsto g \circ C_k(f, R). \end{aligned}$$

The corresponding maps on (co)homology are denoted by $H_*(f, R)$ and $H^*(f, R)$ respectively. This turns singular (co)homology into a functor. $\diamond$

More details on singular (co)homology can be found in any textbook on algebraic topology.

1.2 ■ Definition of the simplicial volume

We will first introduce a norm on the singular chain complex with real coefficients, the ℓ^1-norm. This norm induces a seminorm on singular homology.

The simplicial volume of an oriented closed connected manifold is then defined as the seminorm of the fundamental homology class.

Definition (1.6). Let X be a topological space, and $k \in \mathbf{N}$. For all singular chains $a = \sum_{\sigma \in S_k(X)} a_\sigma \cdot \sigma \in C_k(X, \mathbf{R})$ we define

$$\|a\|_1 := \sum_{\sigma \in S_k(X)} |a_\sigma|. \qquad \diamond$$

Since the sum is finite, this norm is well-defined and turns the singular chain group $C_k(X, \mathbf{R})$ into a normed real vector space.

Definition (1.7). Let X be a topological space, $k \in \mathbf{N}$ and $\alpha \in H_k(X, \mathbf{R})$. Then $\|\cdot\|_1$ induces a seminorm as follows:

$$\|\alpha\|_1 := \inf\{\|a\|_1 \mid a \in C_k(X, \mathbf{R}), \partial(a) = 0, [a] = \alpha\}. \qquad \diamond$$

In general, this is not a norm since the set of boundaries does not need to be closed in the ℓ^1-norm. Geometrically, this seminorm measures how efficiently a homology class can be represented by singular chains.

The fundamental homology class of an n-dimensional oriented closed connected manifold M is the generator of $H_n(M) \cong \mathbf{Z}$ which is compatible with the orientation of M [Massey, Theorem XIV 2.2].

Definition (1.8). Let M be an oriented closed connected n-dimensional manifold. We will denote the corresponding fundamental homology class of M by $[M] \in H_n(M)$. Moreover, we write $[M]_{\mathbf{R}}$ for the image of $[M]$ under the change of coefficients homomorphism $H_n(M, \mathbf{Z}) \longrightarrow H_n(M, \mathbf{R})$. $\diamond$

Definition (1.9). The **simplicial volume** of an oriented closed connected manifold M is defined as

$$\|M\| := \big\|[M]_{\mathbf{R}}\big\|_1. \qquad \diamond$$

Since $\|\cdot\|_1$ is not affected by multiplication with -1, the simplicial volume is independent of the chosen orientation.

A good geometric description of $[M]$ can be given in terms of triangulations [Bredon, page 338f]: if M is triangulable and $\sigma_1, \dots, \sigma_m \colon \Delta^n \longrightarrow M$ is a triangulation of M, then $\sum_{j=1}^m \sigma_j$ represents $[M]$. Hence the simplicial volume measures in some sense the complexity of M.

Of course, it is also possible to use other seminorms on singular homology (to get different "simplicial volumes") or to generalise the above definition to a larger class of spaces. We will sketch such extensions in Section 1.5.

The following lemma indicates why the simplicial volume is akin to a volume (for the simplicial volume, the simplices are all given the weight 1 instead of the integral expression, and only the absolute values of the coefficients are considered).

Lemma (1.10). *Let M be an oriented closed connected Riemannian manifold of dimension n. If $\sum_{\sigma \in S_n(M)} a_\sigma \cdot \sigma \in C_n(M, \mathbf{R})$ is a cycle representing $[M]_{\mathbf{R}}$ consisting of smooth singular simplices, then*

$$\mathrm{vol}(M) = \sum_{\sigma \in S_n(M)} a_\sigma \cdot \int_\sigma \mathrm{vol}_M .$$

Here vol_M denotes the volume form of M and the integral is explained in Definition (4.29).

Proof. Since M is smooth, there is a triangulation $\sigma_1, \dots, \sigma_m \colon \Delta^n \longrightarrow M$ [Whitehead]. Then one shows that the lemma is true for the cycle $\sum_{j=1}^m \sigma_j$ (which is intuitively clear) and uses the fact that the integral only depends on the represented homology class [Lee$_2$, Exercise 16.2 and Theorem 16.10]. ☐

Stronger connections – such as the proportionality principle – between the simplicial volume and other volumes are discussed in Subsection 1.3.2 and Chapter 5.

As a first example we will give a direct calculation for $\|S^1\|$. There are much more sophisticated techniques for computing $\|S^1\|$, but the following proof emphasises geometric intuition.

Theorem (1.11). *Let X be an arcwise connected topological space. Then $\|\alpha\|_1 = 0$ holds for all $\alpha \in H_1(X, \mathbf{R})$.*

Proof. The universal coefficient theorem yields

$$H_1(X) \otimes \mathbf{R} \cong H_1(X, \mathbf{R}).$$

In other words: if $\alpha \in H_1(X, \mathbf{R})$, then there exist $k \in \mathbf{N}$, $r_1, \dots, r_k \in \mathbf{R}$, and $\alpha_1, \dots, \alpha_k \in H_1(X)$ such that α corresponds to

$$\sum_{j=1}^{k} r_j \cdot \alpha_j.$$

By the Hurewicz theorem each of the α_j can be represented by a continuous function $\sigma_j \colon S^1 \longrightarrow X$. For each $j \in \{1, \ldots, k\}$ and each $d \in \mathbf{N}_{\geq 1}$ we write $\sigma_j^{(d)}$ for the composition ("wrapping σ_j around the circle d times")

$$\sigma_j \circ (z \mapsto z^d) \colon S^1 \longrightarrow X.$$

Now the Hurewicz theorem tells us that $[\sigma_j^{(d)}] = d \cdot [\sigma_j]$ holds in $H_1(X)$ (and thus also in $H_1(X, \mathbf{R})$). In particular,

$$\|\alpha\|_1 \leq \left\| \sum_{j=1}^{k} r_j \cdot \frac{\sigma_j^{(d)}}{d} \right\|_1 \leq \frac{1}{d} \cdot \sum_{j=1}^{k} |r_j|.$$

Since this is true for any $d \in \mathbf{N}_{\geq 1}$, we get $\|\alpha\|_1 = 0$. □

Corollary (1.12). *In particular,* $\|S^1\| = 0$. □

This corollary also implies that the seminorm $\|\cdot\|_1$ on $H_1\left(S^1, \mathbf{R}\right)$ is not a norm.

It is more difficult to show that there are manifolds with non-zero simplicial volume. For example, the simplicial volume of hyperbolic manifolds does not vanish (see Theorem (1.23)).

1.3 ■ A zoo of properties

In the first subsection, some elementary properties, such as homotopy invariance and multiplicativity of the simplicial volume are deduced from the definition. We then give a collection of more sophisticated properties, based on the comparison with other volumes and on the theory of bounded cohomology. This section ends with a short (and rather incomplete) discussion of realisability.

1.3.1 ■ Multiplicativity

One of the main characteristics of the simplicial volume is that it is multiplicative with respect to the degree of maps, which enables us to compute the simplicial volume in some special cases.

We start with a simple – but fundamental – observation. Gromov calls the described property **functoriality** [Gromov$_4$, 5.34].

Lemma (1.13). *Let $f\colon X \longrightarrow Y$ be a continuous map of topological spaces, $k \in \mathbf{N}$ and $\alpha \in H_k(X,\mathbf{R})$. Then we get for the induced seminorm*

$$\|H_k(f,\mathbf{R})(\alpha)\|_1 \leq \|\alpha\|_1 .$$

Proof. Let $\sum_{\sigma\in S_k(X)} a_\sigma \cdot \sigma \in C_k(X,\mathbf{R})$ be a cycle representing α. Then the class $H_k(f,\mathbf{R})(\alpha)$ is represented by $\sum_{\sigma\in S_k(X)} a_\sigma \cdot f\circ\sigma$. Therefore

$$\begin{aligned}\|H_k(f,\mathbf{R})(\alpha)\|_1 &= \Bigl\|\Bigl[\sum_{\sigma\in S_k(X)} a_\sigma \cdot f\circ\sigma\Bigr]\Bigr\|_1 \leq \Bigl\|\sum_{\sigma\in S_k(X)} a_\sigma \cdot f\circ\sigma\Bigr\|_1 \\ &\leq \sum_{\sigma\in S_k(X)} |a_\sigma| = \Bigl\|\sum_{\sigma\in S_k(X)} a_\sigma\cdot\sigma\Bigr\|_1 .\end{aligned}$$

Taking the infimum over all cycles representing α yields the desired inequality $\|H_k(f,\mathbf{R})(\alpha)\|_1 \leq \|\alpha\|_1$. □

Multiplicativity (with respect to the degree of maps) is now a direct consequence of this lemma.

Definition (1.14). Let M and N be oriented closed connected n-dimensional manifolds and let $[M] \in H_n(M)$, $[N] \in H_n(N)$ be their fundamental homology classes. Then the **degree** of a continuous map $f\colon M \longrightarrow N$ is defined as the integer $\deg(f)$ satisfying

$$H_n(f)\bigl([M]\bigr) = \deg(f)\cdot[N] . \qquad \diamond$$

Remark (1.15). In the situation of the definition,

$$H_n(f,\mathbf{R})\bigl([M]_\mathbf{R}\bigr) = \deg(f)\cdot[N]_\mathbf{R}$$

also holds (by the universal coefficient theorem). ◇

Lemma (1.16). *Let $f\colon M \longrightarrow N$ be a map of oriented closed connected manifolds of the same dimension. Then*

$$\|M\| \geq |\deg(f)|\cdot\|N\| .$$

Proof. Using the previous lemma and the above remark we conclude that

$$\begin{aligned}\|M\| &= \bigl\|[M]_\mathbf{R}\bigr\|_1 \\ &\geq \bigl\|H_*(f,\mathbf{R})([M]_\mathbf{R})\bigr\|_1 \\ &= \bigl\|\deg(f)\cdot[N]_\mathbf{R}\bigr\|_1 \\ &= |\deg(f)|\cdot\|N\| .\end{aligned}$$

□

Corollary (1.17). *Since homotopy equivalences between oriented closed connected manifolds are maps of degree* 1 *or* -1*, the simplicial volume is a homotopy invariant.* □

Corollary (1.18). *If* $f\colon M \longrightarrow M$ *is a self-map of an oriented closed connected manifold with* $|\deg(f)| \geq 2$*, then*

$$\|M\| = 0.$$

In particular, $\|S^n\| = 0$ *for all* $n \in \mathbf{N}_{\geq 1}$ *and* $\|S^1 \times S^1\| = 0$. □

Corollary (1.19). *Let* $p\colon M \longrightarrow N$ *be a* d*-sheeted covering of oriented closed connected manifolds. Then*

$$\|M\| = d \cdot \|N\|.$$

Proof. Let $n := \dim N$. Since p is a local homeomorphism, it also follows that $\dim M = n$. Of course, we may assume that p is orientation preserving. Thus $\deg(p)$ is well-defined and [Dold, Proposition VIII.4.7]

$$\deg(p) = d.$$

Hence, $\|M\| \geq d \cdot \|N\|$. As second step, we have to prove the "$\leq$" direction: For $\sigma \in S_n(N)$, let $L(\sigma)$ be the set of singular simplices $\widetilde{\sigma} \in S_n(M)$ with $p \circ \widetilde{\sigma} = \sigma$. Since p is a d-sheeted covering and Δ^n is arcwise connected and simply connected, the set $L(\sigma)$ contains precisely d elements. If $a = \sum_{\sigma \in S_n(N)} a_\sigma \cdot \sigma \in C_n(N, \mathbf{R})$ represents $[N]_{\mathbf{R}}$, then

$$\widetilde{a} := \sum_{\sigma \in S_n(N)} a_\sigma \cdot \sum_{\widetilde{\sigma} \in L(\sigma)} \widetilde{\sigma}$$

represents $[M]_{\mathbf{R}}$ because $\partial(\widetilde{a}) = 0$,

$$\begin{aligned} H_n(p, \mathbf{R})\big([\widetilde{a}]\big) &= \Big[\sum_{\sigma \in S_n(N)} a_\sigma \cdot d \cdot \sigma\Big] = d \cdot [N]_{\mathbf{R}} \\ &= H_n(p, \mathbf{R})\big([M]_{\mathbf{R}}\big), \end{aligned}$$

and $\deg(p) = d \neq 0$. Thus

$$\|M\| = \big\|[M]_{\mathbf{R}}\big\|_1 \leq \|\widetilde{a}\|_1 \leq d \cdot \sum_{\sigma \in S_n(N)} |a_\sigma| = d \cdot \|a\|_1,$$

which implies $\|M\| \leq d \cdot \big\|[N]_{\mathbf{R}}\big\|_1 = d \cdot \|N\|$. □

Using this result we can apply covering theory to obtain an upper bound for the simplicial volume of oriented closed connected surfaces [Gromov$_2$, page 217], [Benedetti, Petronio, Proposition C.4.7]:

Theorem (1.20). *Let M be an oriented closed connected surface of genus $g \in \mathbf{N}_{\geq 2}$. Then*

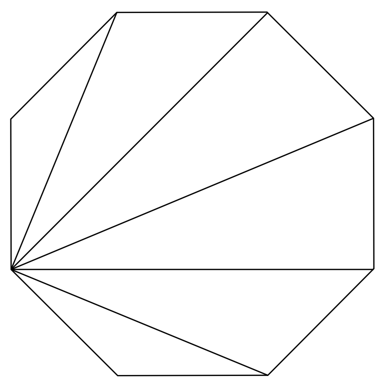
$$\|M\| \leq 4g - 4.$$

To prove this theorem we need the following (slightly weaker) inequality:

Lemma (1.21). *Let $g \in \mathbf{N}_{\geq 2}$ and let F_g be the oriented closed connected surface of genus g. Then*

$$\|F_g\| \leq 4g - 2.$$

Proof. The surface F_g can be constructed from a regular $4g$-gon by identifying certain edges [Massey, §7 of Chapter I]. This $4g$-gon can obviously be triangulated by $4g-2$ simplices:



This triangulation induces a "triangulation" of F_g consisting of $4g-2$ simplices $\sigma_1, \dots, \sigma_{4g-2}$. This is no triangulation of F_g in the strict sense, but its second barycentric subdivision is. Hence we get [Bredon, page 338f] that $\sum_{j=1}^{4g-2} \sigma_j$ is a cycle representing $[F_g]_{\mathbf{R}}$, which yields the inequality

$$\|F_g\| \leq 4g - 2. \qquad \square$$

One can now combine the above inequality with a beautiful covering theory argument to prove the theorem.

Proof (of Theorem (1.20)). For each $d \in \mathbf{N}_{\geq 2}$ there is a d-sheeted covering

$$p_d \colon M_d \longrightarrow M$$

with connected covering space: we have [Massey, Example IV 5.3]

$$\pi_1(F_g)^{\mathrm{ab}} \cong \mathbf{Z}^{2g},$$

where $\pi_1(F_g)^{\mathrm{ab}}$ is the abelianisation of $\pi_1(F_g)$. Thus there is a surjective group homomorphism

$$\pi_1(M) \cong \pi_1(F_g) \longrightarrow \pi_1(F_g)^{\mathrm{ab}} \cong \mathbf{Z}^{2g} \longrightarrow \mathbf{Z}.$$

The inverse image under this homomorphism of a subgroup of $\mathbf{Z}$ of index d is a subgroup of $\pi_1(M)$ of index d. Therefore M admits a d-sheeted covering $p_d \colon M_d \longrightarrow M$ (the covering corresponding to this subgroup) [Massey, Corollary V 7.5 and Theorem V 10.2].

Hence M_d is also an oriented closed connected surface. As one can see by using a CW-decomposition (or by calculating the volume if the manifolds are hyperbolic),

$$\chi(M_d) = d \cdot \chi(M).$$

Thus we get for the genus

$$\begin{aligned} g(M_d) &= \frac{-\chi(M_d) + 2}{2} = \frac{-d \cdot \chi(M) + 2}{2} \\ &= d \cdot g - d + 1, \end{aligned}$$

i.e., M_d and $F_{d \cdot g - d + 1}$ are homotopy equivalent [Massey, Theorem I 8.2]. Applying Corollary (1.19) and Lemma (1.21) results in

$$\begin{aligned} \|M\| &= \frac{1}{d} \cdot \|M_d\| = \frac{1}{d} \cdot \big\|F_{g(M_d)}\big\| \\ &\leq \frac{1}{d} \cdot \big(4g(M_d) - 2\big) \\ &= 4g - 4 + \frac{2}{d}. \end{aligned}$$

This inequality holds for all $d \in \mathbf{N}_{\geq 2}$. Therefore it follows that

$$\|M\| \leq 4g - 4. \qquad \square$$

The estimate of Theorem (1.20) is in fact an equality (Example (1.24)).

Additionally, one can use Corollary (1.19) to prove a toy version of the proportionality principle (Theorem (5.1)):

Lemma (1.22). *Let M and N be two oriented closed connected Riemannian manifolds. Suppose there is an oriented closed connected Riemannian manifold U admitting locally isometric covering maps $U \longrightarrow M$ and $U \longrightarrow N$, both with finitely many sheets. Then*
$$\frac{\|M\|}{\operatorname{vol}(M)} = \frac{\|N\|}{\operatorname{vol}(N)}.$$

Proof. In this situation, both the (Riemannian) volume and the simplicial volume are multiplicative with respect to these coverings (Corollary (1.19) and [Lee_2, Exercise 14.5]). □

A proof of the general case requires much more effort. It will be given in Chapter 5, based on the results about measure homology in Chapters 3 and 4.

1.3.2 ■ Other volumes

There is a great variety of connections between the simplicial volume and other volumes arising in differential geometry. We have already seen a distant relation between the simplicial volume and the Riemannian volume in Lemma (1.10). Gromov used the following fact about hyperbolic manifolds in his proof of Mostow rigidity:

Theorem (1.23). *For $n \in \mathbf{N}$ define $v_n \in \mathbf{R}_{>0}$ as the maximal volume of an ideal n-simplex in n-dimensional hyperbolic space. Then for each hyperbolic oriented closed connected manifold M of dimension n,*
$$\frac{\|M\|}{\operatorname{vol}(M)} = \frac{1}{v_n}.$$

Proof. The estimate $\frac{\|M\|}{\operatorname{vol} M} \geq \frac{1}{v_n}$ can be shown by using the so-called straightening of simplices [Thurston, Corollary 6.1.7], [Ratcliffe, Lemma 3 in §11.4]. It is possible to use the smearing technique (which will be explained in Section 5.4) to prove the reverse inequality [Thurston, Theorem 6.2], [Gromov_2, page 235]. But there are also more direct proofs [Ratcliffe, Theorem 11.4.3]. □

In particular, there are oriented closed connected manifolds whose simplicial volume is not zero since $v_n < \infty$ [Benedetti, Petronio, Lemma C.2.3].

Example (1.24). Let M be an oriented closed connected surface of genus $g \in \mathbf{N}_{\geq 2}$. In view of the homotopy invariance of the simplicial volume we may assume that M is hyperbolic (since there is a hyperbolic model of the surface with genus g – a good description of the hyperbolic structure on surfaces is given in the book of Benedetti and Petronio [Benedetti, Petronio, Section B.3]). Hence the Gauß-Bonnet formula yields

$$\mathrm{vol}(M) = 2\pi \cdot \chi(M) = 2\pi \cdot (2g - 2).$$

Moreover, $v_2 = \pi$ [Benedetti, Petronio, Lemma C.2.3]. Thus Theorem (1.23) implies

$$\|M\| = 4g - 4.$$

In particular, the simplicial volume can grow arbitrarily large and it is no bordism invariant. ■

Based on straightening, Inoue and Yano found the following generalisation of Theorem (1.23) [Inoue, Yano]:

Theorem (1.25). *For each $n \in \mathbf{N}$ there is a constant $C_n \in \mathbf{R}_{>0}$ such that the following property holds: Let $\delta \in \mathbf{R}_{>0}$ and let M be an n-dimensional oriented closed connected Riemannian manifold whose sectional curvature is bounded from above by $-\delta$. Then*

$$\|M\| \geq \; C_n \cdot \delta^{n/2} \cdot \mathrm{vol}(M).$$

The proportionality principle can also be seen as a generalisation of Theorem (1.23). Chapters 2, 3 and 4 provide the tools necessary for the proof (based on Thurston's ideas) of this theorem in Chapter 5.

Theorem (1.26) (Proportionality Principle of Simplicial Volume). *Let M and N be oriented closed connected Riemannian manifolds with isometrically isomorphic universal Riemannian coverings. Then*

$$\frac{\|M\|}{\mathrm{vol}(M)} = \frac{\|N\|}{\mathrm{vol}(N)}.$$

Gromov showed that the simplicial volume also can be seen as a topological obstruction for the existence of certain Riemannian metrics [Gromov$_2$, Section 0.5]:

Definition (1.27). Let M be a smooth manifold. Then its **minimal volume**, $\mathrm{minvol}(M)$, is the infimum over all volumes $\mathrm{vol}_g(M)$, where g is a complete Riemannian metric on M such that the sectional curvature satisfies $|\sec_g(M)| \leq 1$. ◇

Example (1.28). Using the Gauß-Bonnet formula, one can show that the minimal volume of the oriented closed connected surface of genus $g \in \mathbf{N}_{\geq 2}$ is $2\pi \cdot |2 - 2g|$ [Gromov$_2$, page 213].

Scaling Riemannian metrics yields $\operatorname{minvol}(M) = 0$ for all flat manifolds.

A rather counterintuitive result is that $\operatorname{minvol}(S^3) = 0$, as was found out by Berger [Gromov$_2$, page 215]. ■

According to Gromov, the minimal volume can be estimated from below as follows [Gromov$_2$, Section 0.5]:

Theorem (1.29). *For all n-dimensional oriented closed connected smooth manifolds M, the simplicial volume is bounded from above by the minimal volume via*

$$\|M\| \leq (n-1)^n \cdot n! \cdot \operatorname{minvol}(M).$$

Corollary (1.30). *In particular, if M is an oriented closed connected flat manifold, then $\|M\| = 0$.* □

We will prove this corollary as an application of the proportionality principle in Section 5.6, without making use of Theorem (1.29).

1.3.3 ■ A dual point of view

Gromov also discovered that bounded cohomology can be used to investigate the simplicial volume [Gromov$_2$, Chapter 1]. We will sketch this approach in Chapter 2. One of the main results is that the simplicial volume can be computed in terms of bounded cohomology of groups (Corollary (2.32)). This has the following rather striking consequences:

Theorem (1.31).
1. *If M is an oriented closed connected manifold with amenable fundamental group, then $\|M\| = 0$.*
2. *If $f \colon M \longrightarrow N$ is a map of degree 1 between oriented closed connected manifolds of the same dimension such that $\ker \pi_1(f)$ is amenable, then*

$$\|M\| = \|N\|.$$

The class of amenable groups contains, for example, all Abelian groups and all finite groups, but it does not contain the free product $\mathbf{Z} * \mathbf{Z}$. In particular, we see again that $\|S^n\| = 0$ for all $n \in \mathbf{N}_{>0}$ and that $\|S^1 \times S^1\| = 0$. More details will be given in Sections 2.3.3 and 2.4.3.

Other consequences of the duality between $\|\cdot\|_1$ and bounded cohomology are (Theorem (2.14) and Theorem (2.35)):

Theorem (1.32). 1. *For each $n \in \mathbf{N}_{>0}$ there is a constant $c(n) \in \mathbf{R}_{>0}$ such that the following holds: if M and N are two oriented closed connected manifolds, then*

$$\|N\| \cdot \|M\| \leq \|N \times M\| \leq c(\dim N + \dim M) \cdot \|N\| \cdot \|M\| .$$

2. *If M and N are two oriented closed connected manifolds of the same dimension $n \geq 3$, then*

$$\|M \# N\| = \|M\| + \|N\| .$$

In particular, for all oriented closed connected manifolds M, the simplicial volume of the product $S^1 \times M$ is zero. This is generalised in the following way [Yano]:

Theorem (1.33). *Suppose M is an oriented closed connected manifold admitting a non-trivial S^1-action. Then $\|M\| = 0$.*

1.3.4 ■ Which simplicial volumes are possible?

In this paragraph we want to analyse which values occur as the simplicial volume of some oriented closed connected manifold.

For each $v \in \mathbf{R}_{>0}$ there is an oriented closed connected Riemannian manifold M such that $\operatorname{vol}(M) = v$. A very clumsy estimate, based on the following theorem, shows that this cannot be true for the simplicial volume:

Theorem (1.34). *There are only countably many homotopy types CW-complexes consisting of finitely many cells.*

Proof. We prove by induction (on $k \in \mathbf{N}$) that there are only countably many homotopy types of CW-complexes having at most k cells.

If $k = 0$, there is nothing to show. For the induction step let $k \in \mathbf{N}$, and let Y be a CW-complex consisting of $k + 1$ cells. Then there is a CW-complex X with k cells and an attaching map $f\colon S^n \longrightarrow X$ for some $n \in \mathbf{N}$ such that

$$Y \cong X \cup_f D^{n+1}.$$

Here the homotopy type of Y depends only on the homotopy class of f and the homotopy type of X [Lundell, Weingram, Corollary IV 2.4].

By induction, there are only countably many homotopy types of such X. Hence we have to count the number of homotopy classes of possible attaching maps. Let $s \in S^n$ and $x \in X$ be two base points. If x is in the same connected component as $f(S^n)$, then f is homotopic to a pointed map $S^n \longrightarrow X$ because $\{s\} \hookrightarrow S^n$ is a cofibration. But there are only countably many homotopy classes of pointed maps $S^n \longrightarrow X$ because $\pi_n(X, x)$ is countable [Lundell, Weingram, Theorem IV 6.1]. Moreover, the finite CW-complex X can only have finitely many connected components, and there are, of course, only countably many possible dimensions $n \in \mathbf{N}$.

Hence there can be only countably many homotopy types of CW-complexes with $k + 1$ cells. □

Corollary (1.35). *The set*

$$\big\{\|M\| \bigm| M \text{ is an oriented closed connected manifold}\big\} \subset \mathbf{R}_{\geq 0}$$

is countable. In particular, not every positive real number can be realised as simplicial volume of some oriented closed connected manifold.

Proof. Each closed (oriented, connected) topological manifold has the homotopy type of a finite CW-complex [Kirby, Siebenmann, page 744]. By the previous theorem, there are only countably many homotopy types of closed (oriented, connected) manifolds. Now Corollary (1.17) completes the proof. □

This corollary shows that the simplicial volume has a much more subtle behaviour than the Riemannian volume: the simplicial volume cannot be affected by "scaling."

The corollary gives, of course, a very unsatisfying and rough answer. It would be interesting to know whether the simplicial volumes lie densely in $\mathbf{R}_{>0}$. It would be sufficient to know that there is a sequence $(M_n)_{n\in\mathbf{N}}$ of oriented closed connected manifolds (of dimension at least 3) with non-zero simplicial volume such that

$$\lim_{n\to\infty} \|M_n\| = 0.$$

Then one could use Gromov's formula for connected sums (Theorem (1.32)) to see that the simplicial volumes lie dense in $\mathbf{R}_{>0}$.

Example (1.24) shows at least that the simplicial volume can become arbitrarily large.

The question of realisability has already been posed by Gromov [Gromov$_2$, page 222]. He suspects that the simplicial volumes might form a closed well-ordered (with the standard ordering on the real line) non-discrete subset of $\mathbf{R}$. This speculation is based on a result of Thurston on hyperbolic manifolds of dimension 3 [Gromov$_1$], [Thurston]. However, until today there do not seem to exist any detailed answers to all these questions.

1.4 ■ Simplicial volume and L^2-invariants

As mentioned in the introduction, there seems to be a deep connection between the simplicial volume and L^2-invariants, such as L^2-Betti numbers and L^2-torsion. More concretely, Gromov formulated the following conjecture [Gromov$_3$, Section 8A]:

Conjecture. *Let M be an aspherical closed orientable manifold whose simplicial volume vanishes. Then for all $p \in \mathbf{N}$,*

$$b_p^{(2)}\big(\widetilde{M};\mathcal{N}(\pi_1(M))\big) = 0.$$

Lück extended this conjecture as follows [Lück$_1$, Conjecture 3.2]:

Moreover, $\widetilde{M}$ is of determinant class and $\varrho^{(2)}\big(\widetilde{M};\mathcal{N}(\pi_1(M))\big) = 0$.

Since aspherical spaces are the central objects of concern, we recall the definition:

Definition (1.36). A space is called **aspherical** if its universal covering is contractible. (For CW-complexes this is the same as to require that all higher homotopy groups vanish). ◇

The L^2-Betti numbers $b_*^{(2)}$ and L^2-torsion $\varrho^{(2)}$ can be defined (in the setting of von Neumann algebras) similarly to their classic pendants [Lück$_2$, Chapter 6].

Until now there does not seem to exist a strategy to prove (or disprove) the above conjecture. However, there are many facts witnessing the probable truth of this conjecture. In the following paragraphs we will present some of this evidence. A more extensive list can be found in Lück's book [Lück$_2$, Chapter 14]. For simplicity we refer to the proofs in this book instead to the original papers.

1.4.1 ■ The zeroth L^2-Betti number

A rather easy argument shows that the above conjecture holds for the zeroth L^2-Betti number:

Lemma (1.37). *Let M be an aspherical connected closed manifold. Then the fundamental group $\pi_1(M)$ is torsion free and hence infinite.*

Proof. Assume that $\pi_1(M)$ contains a finite cyclic subgroup G. Since M is aspherical and of finite dimension n, the cellular chain complex $C_*^{\text{cell}}(\widetilde{M};\mathbf{Z})$ is a bounded projective resolution of the trivial $\mathbf{Z}G$-module $\mathbf{Z}$. This implies that the group homology

$$H_*(G;\mathbf{Z}) \cong H_*\big(C_*^{\text{cell}}(\widetilde{M};\mathbf{Z}) \otimes_{\mathbf{Z}G} \mathbf{Z}\big)$$

vanishes in all degrees greater than n.

On the other hand, it is well-known that the group homology of nontrivial finite cyclic groups does not have this property [Weibel, Theorem 6.2.3]. Hence G must be trivial, i.e., $\pi_1(M)$ is torsion free. □

Corollary (1.38). *If M is an aspherical closed connected manifold, then [Lück$_2$, Theorem 6.54(8)]*

$$b_0^{(2)}\big(\widetilde{M};\mathcal{N}(\pi_1(M))\big) = \frac{1}{|\pi_1(M)|} = 0. \qquad \square$$

Example (1.39). The condition that the manifold be aspherical in Gromov's conjecture is necessary: The manifold S^2 is closed and orientable, but not aspherical because $\pi_2(S^2) \cong \mathbf{Z}$. By Corollary (1.18) we have $\|S^2\| = 0$. On the other hand [Lück$_2$, Theorem 6.54(8)],

$$b_0^{(2)}\big(\widetilde{S^2};\mathcal{N}(\pi_1(S^2))\big) = \frac{1}{|\pi_1(S^2)|} = 1 \neq 0. \qquad \blacksquare$$

1.4.2 ■ Amenability

If M is on oriented closed connected (not necessarily aspherical) manifold with amenable fundamental group, then $\|M\| = 0$ (Corollary (2.32)). The corresponding result in the L^2-world is:

Theorem (1.40). *Let M be an oriented closed connected aspherical manifold with amenable fundamental group. Then all L^2-Betti numbers of M are zero.*

Proof. By Lemma (1.37), the fundamental group of M must be infinite. But then it can be shown that all L^2-Betti numbers of M must be zero [Lück$_2$, Corollary 6.75]. □

1.4.3 ■ Proportionality

One of the most remarkable similarities between the simplicial volume and L^2-invariants is the proportionality principle. It is valid for the simplicial volume (Theorem (5.1)) and for L^2-Betti numbers (and L^2-torsion) [Lück$_2$, Theorem 3.183]:

Theorem (1.41) (Proportionality Principle of L^2-Invariants). *Let U be a simply connected Riemannian manifold. Then for each $p \in \mathbf{N}$ there are constants $B_p \in \mathbf{R}_{>0}$ and a constant $T \in \mathbf{R}_{>0}$ satisfying: for any discrete group G acting freely and cocompactly on U by isometries and all $p \in \mathbf{N}$,*

$$b_p^{(2)}\bigl(U;\mathcal{N}(G)\bigr) = B_p \cdot \operatorname{vol}(G \setminus U).$$

If U with this G-action is of determinant class, then

$$\varrho^{(2)}\bigl(U;\mathcal{N}(G)\bigr) = T \cdot \operatorname{vol}(G \setminus U).$$

1.4.4 ■ Multiplicativity

As we have seen in Section 1.3, the simplicial volume is multiplicative with respect to the mapping degree. Therefore Gromov's conjecture predicts that if M is an aspherical oriented closed connected manifold admitting a self-map of degree at least two, then all L^2-Betti numbers of M vanish. This statement is known to be true if this self-map is a covering [Lück$_2$, Example 1.37].

The general case is still open, but there is at least a partial result:

Definition (1.42). A group G is said to be **Hopfian** if every epimorphism $G \longrightarrow G$ is an isomorphism. ◇

Examples for Hopfian groups are finite groups, the integers, and, more generally, residually finite groups. The group $\ell^\infty(\mathbf{N}, \mathbf{R})$ is not Hopfian as can be seen by considering the shift operator.

Theorem (1.43). *Let M be an aspherical oriented closed connected manifold. If there is a continuous map $f\colon M \longrightarrow M$ such that $|\deg(f)| \geq 2$ and if all subgroups of $\pi_1(M)$ of finite index are Hopfian, then*

$$\forall_{p \in \mathbf{N}} \quad b_p^{(2)}\big(\widetilde{M}; \mathcal{N}(\pi_1(M))\big) = 0.$$

A proof is given in [Lück$_2$, Theorem 14.40]. (There only all *normal* subgroups of $\pi_1(M)$ are required to be Hopfian – but this is a misprint.)

1.4.5 ■ Non-trivial S^1-operations

Furthermore, the simplicial volume and the L^2-Betti numbers behave similarly under S^1-operations [Lück$_2$, Corollary 1.43]:

Theorem (1.44). *Let M be an aspherical closed smooth manifold with non-trivial S^1-action. Then*

$$\forall_{p \in \mathbf{N}} \quad b_p^{(2)}\big(\widetilde{M}; \mathcal{N}(\pi_1(M))\big) = 0.$$

The corresponding statement about the simplicial volume is Theorem (1.33).

1.4.6 ■ Gromov's conjecture

Looking at the conjecture and the known evidence one could also suspect that the converse is true: If M is an aspherical oriented closed connected manifold whose L^2-Betti numbers vanish, then $\|M\| = 0$.

But this statement is false: If M is an oriented closed connected hyperbolic manifold of odd dimension, then

$$b_p^{(2)}\big(\widetilde{M}; \mathcal{N}(\pi_1(M))\big) = 0$$

for all $p \in \mathbf{N}$ [Lück$_2$, Theorem 1.54]. On the other hand, $\|M\| \neq 0$ by Theorem (1.23).

1.5 ■ Generalisations

In the following, some possible generalisations of the simplicial volume are described. In the first part, we consider the use of coefficients other than **R**. Later we sketch extensions of the simplicial volume to non-compact, non-connected, or non-orientable manifolds possibly with non-empty boundary, and also to Poincaré complexes. In the last section we consider functorial seminorms in the sense of Gromov.

1.5.1 ■ Unreal coefficients

Instead of real coefficients we could also use the singular chain complex with integral, complex or rational coefficients to define "simplicial volumes" $\|M\|_{\mathbf{Z}}$, $\|M\|_{\mathbf{C}}$, and $\|M\|_{\mathbf{Q}}$.

Definition (1.45). Let $R \in \{\mathbf{Z}, \mathbf{C}, \mathbf{Q}\}$ and let M be an oriented closed connected manifold of dimension n. We write $[M]_R \in H_n(M, R) \cong R$ for the image of the fundamental class $[M]$ under the change of coefficients homomorphism.

As in the case of singular homology with real coefficients, we can define an "R-linear norm" on $C_n(M, R)$ via

$$\sum_{\sigma \in S_n(M)} a_\sigma \cdot \sigma \longmapsto \sum_{\sigma \in S_n(M)} |a_\sigma|,$$

which induces a "seminorm" $\|\cdot\|_1^R$ on $H_n(M, R)$ by taking the infimum over all representatives. That is, we define

$$\|\alpha\|_1^R := \inf\bigl\{ \|a\|_1^R \bigm| a \in C_n(M, R), \partial a = 0, [a] = \alpha \in H_n(M, R) \bigr\}$$

for all $\alpha \in H_n(M, R)$. In particular, we write

$$\|M\|_{\mathbf{Z}} := \bigl\|[M]_{\mathbf{Z}}\bigr\|_1^{\mathbf{Z}}, \quad \|M\|_{\mathbf{C}} := \bigl\|[M]_{\mathbf{C}}\bigr\|_1^{\mathbf{C}}, \quad \|M\|_{\mathbf{Q}} := \bigl\|[M]_{\mathbf{Q}}\bigr\|_1^{\mathbf{Q}}. \qquad \diamond$$

Of course, we could also define a notion of volume using other homology theories based on chain complexes which can be equipped with some "norm" and whose top homology group (of oriented closed connected manifolds) contains a distinguished class.

The simplicial volume with integral coefficients obviously never vanishes. So it cannot be the same as the usual simplicial volume. But the integral simplicial volume has an interesting connection with the Betti numbers of the corresponding manifold. However, the integral version is not very comfortable to work with.

Lemma (1.46). *Suppose M is an oriented closed connected n-dimensional manifold and $\sum_{j=1}^{t} a_j \cdot \sigma_j \in C_n(M, \mathbf{Z})$ represents $[M]_{\mathbf{Z}}$. Then*

$$\sum_{k \in \mathbf{N}} b_k(M) \leq t \cdot 2^{n+1},$$

where $b_k(M) = \operatorname{rk} H_k(M)$ is the k-th Betti number of M (which is the same as $\dim_{\mathbf{R}} H_k(M, \mathbf{R})$ and $\dim_{\mathbf{Q}} H_k(M, \mathbf{Q})$). In particular,

$$\sum_{k \in \mathbf{N}} b_k(M) \leq 2^{n+1} \cdot \|M\|_{\mathbf{Z}}.$$

The proof is based on Poincaré duality [Lück$_2$, Example 14.28]. The following result is in the same direction, but is much harder to prove [Gromov$_2$, page 220].

Theorem (1.47). *For all $k \in \mathbf{R}_{\geq 1}$ and all $n \in \mathbf{N}$, there is a constant $C(n,k)$ with the following property: If M is an n-dimensional oriented closed connected Riemannian manifold with $-k_1 \leq \sec(M) \leq k_2$ for some $k_1 \geq k_2 > 0$, then the Betti numbers of M can be estimated via*

$$\sum_{k \in \mathbf{N}} b_k(M) \leq C(n, k_1/k_2) \cdot \|M\|.$$

It is not hard to see that the real, complex, and rational simplicial volume all coincide:

Lemma (1.48). *Let M an oriented closed connected manifold of dimension n. Then*

$$\|M\|_{\mathbf{C}} = \|M\| = \|M\|_{\mathbf{Q}}.$$

Proof. The inclusions $C_*(M, \mathbf{Q}) \subset C_*(M, \mathbf{R}) \subset C_*(M, \mathbf{C})$ yield

$$\|M\|_{\mathbf{C}} \leq \|M\| \leq \|M\|_{\mathbf{Q}}.$$

The inequality $\|M\|_{\mathbb{C}} \geq \|M\|$ can be shown by considering the real part of a representing cycle. The inequality $\|M\| \geq \|M\|_{\mathbb{Q}}$ can be obtained via an approximation of boundaries with real coefficients by boundaries with rational coefficients. In both cases one has to use the fact that the fundamental class with rational or complex coefficients comes from a class in homology with integral coefficients. $\square$

1.5.2 ■ More spaces

It is also possible to define a simplicial volume in more general settings [Gromov$_2$, page 216]. For example, if M is a closed connected manifold which is not orientable, we can define

$$\|M\| := \frac{1}{2} \cdot \|\overline{M}\|,$$

where $\overline{M}$ is the orientation double covering of M.

If M is not connected we can just sum up the simplicial volumes of the connected components.

If M is an oriented connected manifold without boundary which is not compact, we can use the fundamental class of M in locally finite homology to define a (possibly infinite) simplicial volume.

If M is a manifold with boundary, then we can define the simplicial volume as the seminorm of the relative fundamental class $[M, \partial M]$.

Since the definition of the simplicial volume only makes use of the homological properties of manifolds, it is easy to find a definition of simplicial volume for homology manifolds (with respect to real coefficients) or Poincaré complexes, which are some kind of "homotopy manifolds."

All these extensions preserve basic properties, such as homotopy invariance, multiplicativity etc.

1.5.3 ■ Functorial seminorms

Instead of the seminorm in singular homology with real coefficients induced by the ℓ^1-norm, one could also use other seminorms. To ensure that the resulting invariant has useful properties, Gromov introduced the following notion [Gromov$_4$, 5.34]:

Definition (1.49). Suppose for each topological space X a seminorm $\|\cdot\|$ is given on $H_*(X, \mathbf{R})$. Such a seminorm is called **functorial** if for each continuous map $f \colon X \longrightarrow Y$ and all $\alpha \in H_*(X, \mathbf{R})$

$$\|H_*(f, \mathbf{R})(\alpha)\| \leq \|\alpha\|. \qquad \diamond$$

By Lemma (1.13) the seminorm $\|\cdot\|_1$ on homology with real coefficients induced by the ℓ^1-norm on the singular chain complex is functorial. Other examples are given in Gromov's book [Gromov$_4$, pages 302−316].

The proof of Corollary (1.16) of course shows that each functorial seminorm is submultiplicative with respect to the degree of maps.

Definition (1.50). If $\|\cdot\|'$ is a functorial seminorm, one can define an analogue of the simplicial volume as the value of $\|\cdot\|'$ on the fundamental class. $\diamond$

Then the same argument as in Corollary (1.18) shows that $\|S^n\|'$ must be zero for all $n \in \mathbf{N}_{>0}$.

Using bounded cohomology, we will see in Corollary (2.28) that the seminorm induced by the ℓ^1-norm vanishes on simply connected spaces. Gromov conjectured that this phenomenon occurs for all functorial seminorms [Gromov$_4$, Remark in 5.35]:

Conjecture. *Any functorial seminorm on singular homology with real coefficients vanishes for simply connected spaces.*

Until today, neither a proof nor counterexamples to this amazing conjecture are known.

2 Bounded Cohomology

Bounded cohomology is the functional analytic twin of singular cohomology – it is constructed similarly to singular cohomology from the singular chain complex, using the topological dual space instead of the algebraic dual space [Gromov$_2$]. The corresponding duality on the level of (co)homology is explored in Section 2.2. Using this duality, it is possible to calculate the simplicial volume via bounded cohomology.

It is astonishing that the rather small difference between the definitions of singular and bounded cohomology lead to completely different characters of both theories.

Apart from the above geometric definition, there is also a notion of bounded cohomology of groups due to Ivanov based on techniques from homological algebra [Ivanov$_1$], which we will sketch in Section 2.3. This algebraic approach is the source of the strength of bounded cohomology.

In Section 2.4, a key feature of bounded cohomology is discussed: the bounded cohomology of a topological space coincides with the bounded cohomology of its fundamental group. Unfortunately, unlike singular cohomology bounded cohomology fails to satisfy the excision axiom. Nevertheless, in many cases bounded cohomology can be calculated directly from special resolutions. For example, it can be shown that bounded cohomology ignores amenable groups, implying that the bounded cohomology of spaces with amenable fundamental group vanishes.

In the last section, we will explain a special resolution for calculating the bounded cohomology of a group, which plays a crucial rôle in Section 4.3.

2.1 ■ Bounded cohomology of spaces

Bounded cohomology is the functional analytic twin of singular cohomology. It is constructed via the topological dual of the singular chain complex instead of the algebraic one. The corresponding norm for singular cochains is therefore the supremum norm:

Definition (2.1). Let X be a topological space and $k \in \mathbf{N}$. For a singular cochain $f \in C^k(X, \mathbf{R})$ the (possibly infinite) **supremum norm** is defined via

$$\|f\|_\infty := \sup_{\sigma \in S_k(X)} |f(\sigma)|.$$

This induces a seminorm on $H^k(X, \mathbf{R})$ by

$$\forall_{\varphi \in H^k(X,\mathbf{R})} \quad \|\varphi\|_\infty := \inf\bigl\{ \|f\|_\infty \bigm| f \in C^k(X, \mathbf{R}), \delta(f) = 0, [f] = \varphi \bigr\}.$$

We write

$$\widehat{C}^k(X) := \bigl\{ f \in C^k(X, \mathbf{R}) \bigm| \|f\|_\infty < \infty \bigr\}.$$

for the vector space of **bounded k-cochains**. ◇

It is easy to see that the coboundary operator on the singular cochain complex $C^k(X, \mathbf{R})$ restricts to bounded cochains, i.e.,

$$\delta\bigl(\widehat{C}^k(X)\bigr) \subset \widehat{C}^{k+1}(X).$$

Thus $\widehat{C}^*(X)$ is a cochain complex.

Definition (2.2). Let X be a topological space and $k \in \mathbf{N}$. The **k-th bounded cohomology group of X** is defined by

$$\widehat{H}^k(X) := \frac{\ker\bigl(\delta|_{\widehat{C}^k(X)} \colon \widehat{C}^k(X) \longrightarrow \widehat{C}^{k+1}(X)\bigr)}{\operatorname{im}\bigl(\delta|_{\widehat{C}^{k-1}(X)} \colon \widehat{C}^{k-1}(X) \longrightarrow \widehat{C}^k(X)\bigr)}.$$ ◇

As for the simplicial volume, it is not immediately clear from the definition that there are spaces whose bounded cohomology does not vanish. An example of such a space is $S^1 \vee S^1$ (Example (2.29)).

Definition (2.3). Let X be a topological space and $k \in \mathbf{N}$. The supremum norm on $\widehat{C}^k(X)$ induces a seminorm on $\widehat{H}^k(X)$ by

$$\forall_{\varphi \in \widehat{H}^k(X)} \quad \|\varphi\|_\infty := \inf\bigl\{\|f\|_\infty \bigm| f \in \widehat{C}^k(X), \delta(f) = 0, [f] = \varphi\bigr\}. \qquad \diamond$$

Except for some special cases it is not known whether this seminorm is a norm [Ivanov$_2$], [Soma].

2.1.1 ■ Elementary properties of bounded cohomology

As we will see later, bounded cohomology is not as tame as singular cohomology because bounded cohomology fails to satisfy the excision axiom (so the usual divide-and-conquer approach does not work). But at least bounded cohomology is natural, homotopy invariant and fulfils the dimension axiom.

Lemma (2.4) (Naturality). *Let X, Y be topological spaces and $f\colon X \longrightarrow Y$ a continuous map. For each $k \in \mathbf{N}$, the map f induces a homomorphism*

$$\begin{aligned} \widehat{H}^k(f)\colon \widehat{H}^k(Y) &\longrightarrow \widehat{H}^k(X) \\ [g] &\longmapsto \bigl[c \mapsto g(f \circ c)\bigr] \end{aligned}$$

of operator norm at most 1. Here composition of f with elements of $C_k(X, \mathbf{R})$ is defined by linear extension of composition with elements of $S_k(X)$.

Proof. Let $g \in \widehat{C}^k(Y)$ with $\delta(g) = 0$. Since g is bounded on $S_k(Y)$, the cochain

$$\begin{aligned} C_k(X, \mathbf{R}) &\longrightarrow \mathbf{R} \\ c &\longmapsto g(f \circ c) \end{aligned}$$

is bounded on $S_k(X)$. According to the definition of the coboundary operator, this cochain is a cocycle. As the definition of $\widehat{H}^k(f)([g])$ does not depend on the chosen representatives g, the map $\widehat{H}^k(f)$ is indeed well-defined. It is clear that $\widehat{H}^k(f)$ is a homomorphism of real vector spaces and that

$$\bigl\|\widehat{H}^k(f)\bigl([g]\bigr)\bigr\|_\infty \leq \|g\|_\infty.$$

Taking the infimum over all representatives yields

$$\bigl\|\widehat{H}^k(f)\bigl([g]\bigr)\bigr\|_\infty \leq \|[g]\|_\infty. \qquad \square$$

Lemma (2.5) (Homotopy Invariance). *The functor bounded cohomology is homotopy invariant, i.e., if $f, g\colon X \longrightarrow Y$ are homotopic maps of topological spaces, then the induced maps are equal:*

$$\widehat{H}^*(f) = \widehat{H}^*(g)\colon \widehat{H}^*(Y) \longrightarrow \widehat{H}^*(X).$$

Proof. Since f and g are homotopic maps, there is a chain homotopy

$$k_*\colon C_*(X, \mathbf{R}) \longrightarrow C_{*+1}(Y, \mathbf{R})$$

between $C_*(f, \mathbf{R})$ and $C_*(g, \mathbf{R})$ [Bredon, Corollary IV 16.5], i.e.,

$$\partial \circ k_{*+1} + k_* \circ \partial = C_{*+1}(f, \mathbf{R}) - C_{*+1}(g, \mathbf{R}).$$

This chain homotopy induces the map

$$\begin{aligned} h^*\colon C^*(Y, \mathbf{R}) &\longrightarrow C^{*-1}(X, \mathbf{R}) \\ c &\longmapsto (-1)^* \cdot c \circ k_{*-1}. \end{aligned}$$

Obviously, h^* maps bounded cochains to bounded cochains, thus it restricts to

$$\widehat{h}^* := h^*|_{\widehat{C}^*(Y)}\colon \widehat{C}^*(Y) \longrightarrow \widehat{C}^{*-1}(X).$$

Since

$$\begin{aligned} \big(\delta \circ \widehat{h}^* + \widehat{h}^{*+1} \circ \delta\big)(c) &= \delta\big((-1)^* \cdot c \circ k_{*-1}\big) + \widehat{h}^{*+1}\big((-1)^{*+1} \cdot c \circ \partial\big) \\ &= c \circ k_{*-1} \circ \partial + c \circ \partial \circ k_* \\ &= c \circ \big(C_*(f, \mathbf{R}) - C_*(g, \mathbf{R})\big) \\ &= \big(\widehat{C}^*(f) - \widehat{C}^*(g)\big)(c) \end{aligned}$$

holds for all $c \in \widehat{C}^*(Y)$, it follows that $\widehat{h}^*$ is a cochain homotopy between $\widehat{C}^*(f)$ and $\widehat{C}^*(g)$. Hence we get $\widehat{H}^*(f) = \widehat{H}^*(g)$. □

Lemma (2.6) (Dimension Axiom). *Bounded cohomology satisfies the dimension axiom, i.e.,*

$$\begin{aligned} &\widehat{H}^0(\bullet) \cong \mathbf{R}, \text{ and} \\ \forall_{k \in \mathbf{N}_{\geq 1}} \quad &\widehat{H}^k(\bullet) = 0. \end{aligned}$$

Proof. This is the same calculation as in singular cohomology. □

2.2 ■ Duality

In this section the duality between the seminorms $\|\cdot\|_1$ and $\|\cdot\|_\infty$ on singular homology and bounded cohomology respectively is explored. Using the Hahn-Banach theorem we will see that the duality on the level of (co)chain complexes carries over to (bounded co)homology. This duality was discovered and applied by Gromov [Gromov$_2$].

The duality principle (Theorem (2.7)) shows an important aspect of bounded cohomology: bounded cohomology can be used to compute the simplicial volume. Since bounded cohomology is much better understood (in view of the techniques presented in Section 2.3) than the seminorm on homology, duality leads to interesting applications. For example, bounded cohomology can be used to give estimates of the simplicial volume of products and connected sums of manifolds.

Moreover, duality will play a central rôle in the proof that measure homology and singular homology are *isometrically* isomorphic (see Section 4.3).

Theorem (2.7) (Duality Principle). *Let X be a topological space, $k \in \mathbf{N}$ and $\alpha \in H_k(X, \mathbf{R})$.*

1. *Then $\|\alpha\|_1 = 0$ if and only if*
$$\forall_{\varphi \in \widehat{H}^k(X)} \quad \langle \varphi, \alpha \rangle = 0.$$
2. *If $\|\alpha\|_1 > 0$, then*
$$\frac{1}{\|\alpha\|_1} = \inf\bigl\{ \|\varphi\|_\infty \bigm| \varphi \in \widehat{H}^k(X), \langle \varphi, \alpha \rangle = 1 \bigr\}.$$

The theorem makes use of the Kronecker product on bounded cohomology. Therefore, we recall the definition of the Kronecker product on singular (co)homology:

Definition (2.8). Let X be a topological space, R a commutative ring with unit and $k \in \mathbf{N}$. Furthermore, let $f \in C^k(X, R)$ be a cocycle and $a \in C_k(X, R)$ a cycle. Then the **Kronecker product** of $[f] \in H^k(X, R)$ and $[a] \in H_k(X, R)$ is defined by
$$\bigl\langle [f], [a] \bigr\rangle := f(a).$$
◇

In fact, the relations between the coboundary and the boundary operator imply that the Kronecker product is well-defined.

Definition (2.9). Let X be a topological space. The homomorphsim

$$c_X \colon \widehat{H}^*(X) \longrightarrow H^*(X, \mathbf{R})$$

induced by the inclusion $\widehat{C}^*(X) \subset C_*(X, \mathbf{R})$ is called **comparison map**. $\diamond$

In general, the comparison map is neither injective nor surjective: for S^1 it is not injective, for $S^1 \vee S^1$ it cannot be surjective (Corollary (2.28) and Example (2.29)).

Definition (2.10). Let X be a topological space, $k \in \mathbf{N}$, $\varphi \in \widehat{H}^k(X)$, and $\alpha \in H_k(X, \mathbf{R})$. Then the **Kronecker product** of φ and α is defined to be

$$\langle \varphi, \alpha \rangle := \big\langle c_X(\varphi), \alpha \big\rangle. \qquad \diamond$$

Proof (of Theorem (2.7)). Let α be represented by the cycle

$$a = \sum_{\sigma \in S_k(X)} a_\sigma \cdot \sigma \in C_k(X, \mathbf{R}).$$

Assume that there is a $\varphi \in \widehat{H}^k(X)$, represented by the bounded cochain $f \in \widehat{C}^k(X)$, with $\langle \varphi, \alpha \rangle = 1$. Then, by definition,

$$\begin{aligned} 1 = |\langle \varphi, \alpha \rangle| &= \left| f\Big(\sum_{\sigma \in S_k(X)} a_\sigma \cdot \sigma \Big) \right| \leq \sum_{\sigma \in S_k(X)} |a_\sigma| \cdot |f(\sigma)| \\ &\leq \|f\|_\infty \cdot \|a\|_1 . \end{aligned}$$

If we take the infimum over all representatives of φ and α, it follows that

$$1 \leq \|\varphi\|_\infty \cdot \|\alpha\|_1 .$$

In particular, if $\|\alpha\|_1 = 0$, there can be no such φ.

Now, let $\|\alpha\|_1 > 0$, and let $a \in C_k(X, \mathbf{R})$ be a cycle representing α. We denote the closure in $C_k(X, \mathbf{R})$ with respect to the norm $\|\cdot\|_1$ by $\overline{\,\cdot\,}^1$. Thus the condition $\|a\|_1 \geq \|\alpha\|_1 > 0$ implies

$$a \notin \overline{\operatorname{im}\big(\partial \colon C_{k+1}(X, \mathbf{R}) \to C_k(X, \mathbf{R})\big)}^1 .$$

Hence the Hahn-Banach theorem yields the existence of a continuous functional $f\colon C_k(X,\mathbf{R}) \longrightarrow \mathbf{R}$ with

$$\begin{aligned} f(a) &= 1, \\ f|_{\overline{\operatorname{im}\left(\partial\colon C_{k+1}(X,\mathbf{R}) \to C_k(X,\mathbf{R})\right)}^{1}} &= 0, \\ \|f\|_\infty &= \frac{1}{\|a\|_1}. \end{aligned}$$

The second property implies that $f :=\in C^k(X,\mathbf{R})$ is a cocycle. Moreover, $\|f\|_\infty < \infty$, which shows $f \in \widehat{C}^k(X)$. By construction, we have

$$\langle [f], \alpha \rangle = \langle [f], [a] \rangle = f(a) = 1$$

and

$$\|[f]\|_\infty \leq \|f\|_\infty = \frac{1}{\|a\|_1} \leq \frac{1}{\|\alpha\|_1}.$$

In particular, there exists a $\varphi \in \widehat{H}^k(X)$ with $\langle \varphi, \alpha \rangle = 1$ and $\|\varphi\|_\infty \leq 1/\|\alpha\|_1$. Hence, if $\|\alpha\|_1 > 0$,

$$\frac{1}{\|\alpha\|_1} = \inf\left\{ \|\varphi\|_\infty \;\middle|\; \varphi \in \widehat{H}^k(X), \langle \varphi, \alpha \rangle = 1 \right\}.$$ □

Corollary (2.11). *Let $f\colon X \longrightarrow Y$ be a continuous map such that the induced map $\widehat{H}^*(f)\colon \widehat{H}^*(Y) \longrightarrow \widehat{H}^*(X)$ is an isometric isomorphism. Then*

$$\|\alpha\|_1 = \left\| H_*(f,\mathbf{R})(\alpha) \right\|_1$$

for all $\alpha \in H_*(X,\mathbf{R})$.

Proof. Since $\widehat{H}^*(f)$ is an isometric isomorphism, we obtain

$$\begin{aligned} &\left\{ \|\varphi\|_\infty \;\middle|\; \varphi \in \widehat{H}^n(Y), \left\langle \varphi, H_n(f,\mathbf{R})(\alpha) \right\rangle = 1 \right\} \\ &\quad = \left\{ \left\| \widehat{H}^n(f)(\varphi) \right\|_\infty \;\middle|\; \varphi \in \widehat{H}^n(Y), \left\langle \widehat{H}^n(f)(\varphi), \alpha \right\rangle = 1 \right\} \\ &\quad = \left\{ \|\psi\|_\infty \;\middle|\; \psi \in \widehat{H}^n(X), \langle \psi, \alpha \rangle = 1 \right\}. \end{aligned}$$

Therefore application of Theorem (2.7) proves the Corollary. □

2.2.1 ■ Bounded cohomology and simplicial volume

Application of the duality principle to the fundamental class yields a description of the simplicial volume in terms of bounded cohomology:

Corollary (2.12). *Let M be an oriented closed connected n-dimensional manifold.*

1. *Then $\|M\| = 0$ if and only if the comparison homomorphism*
$$c^n_M\colon \widehat{H}^n(M) \longrightarrow H^n(M, \mathbf{R})$$
is trivial.

2. *If $\|M\| > 0$, then*
$$\frac{1}{\|M\|} = \big\|[M]^{\mathbf{R}}\big\|_\infty,$$
where $[M]^{\mathbf{R}} \in H^n(M, \mathbf{R})$ denotes the image of the cohomological fundamental class of M under the change of rings homomorphism.

Proof. By definition, $\big\langle [M]^{\mathbf{R}}, [M]_{\mathbf{R}} \big\rangle = 1$. Furthermore,
$$H^n(M, \mathbf{R}) \cong \mathbf{R} \cdot [M]^{\mathbf{R}}.$$
Hence $\operatorname{im}(c^n_M) \neq 0$ if and only if $[M]^{\mathbf{R}} \in \operatorname{im}(c^n_M)$.

1. Therefore the first part of Theorem (2.7) yields that $\|M\| = \|[M]_{\mathbf{R}}\|_1 = 0$ if and only if c^n_M is trivial.

2. Suppose $\|M\| > 0$. By the first part, $c^n_M\colon \widehat{H}^n(M) \longrightarrow H^n(M, \mathbf{R}) \cong \mathbf{R}$ is not trivial and hence surjective. Moreover,
$$\begin{aligned}
\|\varphi\|_\infty &= \inf\big\{\|f\|_\infty \bigm| f \in \widehat{C}^n(X), \delta(f) = 0, [f] = \varphi \text{ in } \widehat{H}^n(M)\big\} \\
&\geq \inf\big\{\|f\|_\infty \bigm| f \in C^n(X, \mathbf{R}), \delta(f) = 0, [f] = c^n_M(\varphi) \text{ in } H^n(M, \mathbf{R})\big\} \\
&= \|c^n_M(\varphi)\|_\infty
\end{aligned}$$
holds for all $\varphi \in \widehat{H}^n(M)$. Using the surjectivity of c^n_M, we obtain (by the second part of Theorem (2.7))
$$\begin{aligned}
\frac{1}{\|M\|} &= \frac{1}{\big\|[M]_{\mathbf{R}}\big\|_1} \\
&= \inf\big\{\|\varphi\|_\infty \bigm| \varphi \in \widehat{H}^n(M), \langle \varphi, [M]_{\mathbf{R}} \rangle = 1\big\} \\
&\geq \inf\big\{\|c^n_M(\varphi)\|_\infty \bigm| \varphi \in \widehat{H}^n(M), \big\langle c^n_M(\varphi), [M]_{\mathbf{R}} \big\rangle = 1\big\} \\
&\geq \inf\big\{\|\psi\|_\infty \bigm| \psi \in H^n(M, \mathbf{R}) \cong [M]^{\mathbf{R}} \cdot \mathbf{R}, \langle \psi, [M]_{\mathbf{R}} \rangle = 1\big\} \\
&= \big\|[M]^{\mathbf{R}}\big\|_\infty.
\end{aligned}$$

In particular, $\left\|[M]^{\mathbf{R}}\right\|_\infty$ is finite. On the other hand, for each $\varepsilon \in \mathbf{R}_{>0}$ there is a cocycle $f \in C^n(M, \mathbf{R})$ with $[f] = [M]^{\mathbf{R}}$ in $H^n(M, \mathbf{R})$ and

$$\|f\|_\infty \leq \left\|[M]^{\mathbf{R}}\right\|_\infty + \varepsilon < \infty.$$

Thus $f \in \widehat{C}^n(M)$. Let $\varphi \in \widehat{H}^n(M)$ be the corresponding bounded cohomology class. Then

$$\|\varphi\|_\infty \leq \|f\|_\infty \leq \left\|[M]^{\mathbf{R}}\right\|_\infty + \varepsilon$$

and

$$\langle \varphi, [M]_{\mathbf{R}} \rangle = \langle c^n_M(\varphi), [M]_{\mathbf{R}} \rangle = \langle [f], [M]_{\mathbf{R}} \rangle = \left\langle [M]^{\mathbf{R}}, [M]_{\mathbf{R}} \right\rangle = 1,$$

which implies

$$\frac{1}{\|M\|} = \frac{1}{\left\|[M]_{\mathbf{R}}\right\|_1} \leq \|\varphi\|_\infty \leq \left\|[M]^{\mathbf{R}}\right\|_\infty + \varepsilon. \qquad \square$$

In Subsection 2.4.3 we will combine this result with the mapping theorem (Theorem (2.31)) to compute the simplicial volume in some special cases.

2.2.2 ■ The first bounded cohomology group

Using the result of Theorem (1.11) it is not hard to see that the first bounded cohomology group always vanishes:

Corollary (2.13). *For all topological spaces X the group $\widehat{H}^1(X)$ is zero.*

Proof. By Theorem (1.11) the seminorm on $H_1(X, \mathbf{R})$ is trivial if X is arcwise connected. Since the singular homology of a space is the direct sum of the singular homology groups of its path components, this result carries over to all spaces X.

Now let $\varphi \in \widehat{H}^1(X)$ and let $f \in \widehat{C}^1(X)$ be a cocycle representing φ. Since the seminorm on $H_1(X, \mathbf{R})$ is trivial (Theorem (1.11)), we get from Theorem (2.7) that

$$\forall_{\alpha \in H_1(X, \mathbf{R})} \quad 0 = \langle \varphi, \alpha \rangle.$$

By the universal coefficient theorem,

$$\begin{aligned} H^1(X, \mathbf{R}) &\longrightarrow \hom\big(H_1(X), \mathbf{R}\big) \\ \psi &\longmapsto \big(\alpha \mapsto \langle \psi, \alpha \rangle\big) \end{aligned}$$

is an isomorphism. Hence the comparison map $c^1_X\colon \widehat{H}^1(X) \longrightarrow H^1(X,\mathbf{R})$ must be trivial. Thus there is a cochain $u \in C^0(X,\mathbf{R})$ with

$$\delta(u) = f.$$

However, in general u is not bounded. Altering u, we construct a bounded cochain $\widehat{u} \in \widehat{C}^0(X)$ satisfying $\delta(\widehat{u}) = f$: Let $(X)_{i\in I}$ be the path components of X. For each $i \in I$ we choose a point $x_i \in X_i$. If $x \in X$, then there is exactly one $i(x) \in I$ such that $x \in X_{i(x)}$. We define $\widehat{u} \in C^0(X,\mathbf{R})$ to be the linear extension of

$$\forall_{x\in X} \quad \widehat{u}(x) := u(x) - u(x_{i(x)}).$$

Since x and $x_{i(x)}$ are in the same path component, there is a path σ_x in X joining these points. So

$$\begin{aligned} \bigl|\widehat{u}(x)\bigr| &= \bigl|u(\sigma_x(0)) - u(\sigma_x(1))\bigr| \\ &= \bigl|u(\partial(\sigma_x))\bigr| \\ &= \bigl|\delta(u)(\sigma_x)\bigr| \\ &= \bigl|f(\sigma_x)\bigr| \\ &\le \|f\|_\infty, \end{aligned}$$

which implies $\widehat{u} \in \widehat{C}^0(X)$. Furthermore we have for all $\sigma \in S_1(X)$

$$\begin{aligned} \delta(\widehat{u})(\sigma) &= \widehat{u}(\sigma(1) - \sigma(0)) \\ &= \widehat{u}(\sigma(1)) - \widehat{u}(\sigma(0)) \\ &= u(\sigma(1)) - u(x_{i(\sigma(1))}) - u(\sigma(0)) + u(x_{i(\sigma(0))}). \end{aligned}$$

Here $i(\sigma(1)) = i(\sigma(0))$ because $\sigma(1)$ and $\sigma(0)$ lie in the same path component. Hence

$$\begin{aligned} \delta(\widehat{u})(\sigma) &= u(\sigma(1)) - u(\sigma(0)) \\ &= u(-\partial(\sigma)) \\ &= \delta(u)(\sigma) \\ &= f(\sigma). \end{aligned}$$

This proves $\varphi = [f] = 0$ in $\widehat{H}^*(X)$. □

2.2.3 ■ Products of manifolds

Bounded cohomology makes it possible to prove that the simplicial volume is compatible with taking products (up to a constant) [Gromov$_2$, page 218].

Theorem (2.14). *For each $n \in \mathbf{N}_{>0}$ there is a constant $c(n) \in \mathbf{R}_{>0}$ such that the following holds: If M and N are two oriented closed connected manifolds, then*

$$\|N\| \cdot \|M\| \leq \|N \times M\| \leq c(\dim N + \dim M) \cdot \|N\| \cdot \|M\| .$$

The proof makes use of explicit descriptions of the cup-product and the cross-products in singular (co)homology. In order to see the first inequality, duality is applied. A detailed proof is given in the book of Benedetti and Petronio [Benedetti, Petronio, Theorem F.2.5]. The constant $c(n)$ is related to the number of n-simplices needed to triangulate the products $\Delta^p \times \Delta^{n-p}$ for all $p \in \{1, \dots, n\}$.

2.3 ■ Bounded cohomology of groups

In his paper "Foundations of the Theory of Bounded Cohomology," Ivanov developed a transparent derived functor type approach to bounded cohomology [Ivanov$_1$]. Just like Ivanov, we will consider bounded cohomology of discrete groups, but there is also a theory available for topological groups [Monod].

2.3.1 ■ Relatively injective resolutions

Similar to ordinary group cohomology, the bounded cohomology of a group is based upon special kinds of resolutions. The basic objects are relatively injective bounded G-modules.

Definition (2.15). Let G be a discrete group.

- A **bounded G-module** is a Banach space V with a (left) G-action such that $\|g \cdot v\| \leq \|v\|$ for all $g \in G$ and all $v \in V$.
- A **G-morphism** of bounded G-modules is a bounded G-equivariant linear operator.
- An injective G-morphism $i\colon V \longrightarrow W$ between bounded G-modules is said to be **strongly injective** if there is a bounded linear operator

$s\colon W \longrightarrow V$ satisfying $s \circ i = \mathrm{id}$ and $\|s\| \leq 1$ (but s does not need to be G-equivariant).

- A bounded G-module U is **relatively injective** if for any strongly injective G-morphism $i\colon V \longrightarrow W$ and any G-morphism $\alpha\colon V \longrightarrow U$ of bounded G-modules there is a G-morphism $\beta\colon W \longrightarrow U$ of bounded G-modules such that $\beta \circ i = \alpha$ and $\|\beta\| \leq \|\alpha\|$. $\diamond$

Good candidates for relatively injective bounded G-modules are mapping spaces:

Definition (2.16). Let G be a discrete group and let V be a Banach space. We write $B(G, V)$ for the real vector space of bounded functions $G \longrightarrow V$. This vector space admits a left G-action via

$$\forall_{f \in B(G,V)} \quad \forall_{g \in G} \quad g \cdot f := \bigl(h \mapsto f(hg)\bigr). \qquad \diamond$$

Example (2.17). Let V be a Banach space and let G be a discrete group. The supremum norm turns $B(G, V)$ into a bounded G-module. Moreover, this bounded G-module is relatively injective [Ivanov$_1$, Lemma (3.2.2)]. $\blacksquare$

Definition (2.18). Let G be a discrete group. A G**-resolution** of a bounded G-module V is a resolution of V via bounded G-modules and G-morphisms

$$0 \longrightarrow V \longrightarrow V_0 \longrightarrow V_1 \longrightarrow \cdots$$

Such a resolution is **strong** if there is a **contracting homotopy**, i.e., there is a chain contraction $(K_n)_{n \in \mathbf{N}}$ consisting of (not necessarily equivariant) linear operators with $\|K_n\| \leq 1$ for all $n \in \mathbf{N}$. The above resolution is called **relatively injective** if the bounded G-modules $(V_n)_{n \in \mathbf{N}}$ are relatively injective. $\diamond$

Example (2.19). Let G be a discrete group. There always exists a strong relatively injective resolution of the trivial G-module $\mathbf{R}$, the **(inhomogeneous) standard resolution**:

For $n \in \mathbf{N}$ we define $V_n := B(G^n, \mathbf{R})$. Equipped with the supremum norm and the left G-action given by

$$g \cdot f := \bigl((g_1, \dots, g_n) \mapsto f(g_1, \dots, g_{n-1}, g_n \cdot g)\bigr)$$

the module V_n becomes a relatively injective bounded G-module. The homomorphisms

$$\begin{aligned} d_{-1}\colon \mathbf{R} &\longrightarrow V_0 \\ c &\longmapsto (g \mapsto c) \end{aligned}$$

and (for $n \in \mathbf{N}$)

$$\begin{aligned} d_n \colon V_n &\longrightarrow V_{n+1} \\ f &\longmapsto \Big((g_0, \dots, g_{n+1}) \mapsto (-1)^{n+1} \cdot f(g_1, \dots, g_{n+1}) \\ &\qquad\qquad + \sum_{j=1}^{n} (-1)^{n-j} \cdot f(g_0, \dots, g_j g_{j+1}, \dots, g_{n+1}) \Big) \end{aligned}$$

turn $(V_n)_{n \in \mathbf{N}}$ into a strong relatively injective resolution [Ivanov$_1$, (3.4)]. Analogously to the classical case, one can also define the corresponding homogeneous resolution. ■

There is also a version of the fundamental theorem of homological algebra in the setting of bounded G-modules [Ivanov$_1$, Lemma (3.3.2)], [Monod, Lemma 7.2.4 and 7.2.6]:

Theorem (2.20). *Let G be a discrete group and let U and V be bounded G-modules. Suppose*

$$0 \longrightarrow V \xrightarrow{\varepsilon_V} V_0 \xrightarrow{v^0} V_1 \xrightarrow{v^1} \cdots$$

is a strong resolution of V and

$$0 \longrightarrow U \xrightarrow{\varepsilon_U} U_0 \xrightarrow{u^0} U_1 \xrightarrow{u^1} \cdots$$

is a complex of relatively injective bounded G-modules. Then any G-morphism $w \colon V \longrightarrow U$ can be extended to a G-morphism $(w_n)_{n \in \mathbf{N}}$ of complexes:

$$\begin{tikzcd} 0 \arrow[r] & V \arrow[r, "\varepsilon_V"] \arrow[d, "w"] & V_0 \arrow[r, "v^0"] \arrow[d, "w_0"] & V_1 \arrow[r, "v^1"] \arrow[d, "w_1"] & \cdots \\ 0 \arrow[r] & U \arrow[r, "\varepsilon_U"'] & U_0 \arrow[r, "u^0"'] & U_1 \arrow[r, "u^1"'] & \cdots . \end{tikzcd}$$

Any two such extensions are chain homotopic via a G-equivariant homotopy.

2.3.2 ■ Definition of bounded cohomology of groups

In view of Theorem (2.20), bounded cohomology of groups can be defined in the same way as usual group cohomology:

Definition (2.21). If G is a discrete group and V is a bounded G-module, we write V^G for the set of G-fixed points. ◇

Definition (2.22). Let G be a discrete group and let

$$\mathbf{R} \longrightarrow V_0 \xrightarrow{v^0} V_1 \xrightarrow{v^1} \cdots$$

be a strong relatively injective resolution of the trivial bounded G-module $\mathbf{R}$. Then the **bounded cohomology of** G (with coefficients in $\mathbf{R}$), $\widehat{H}^*(G)$, is the cohomology of the induced complex

$$0 \longrightarrow V_0^G \xrightarrow{v^0|} V_1^G \xrightarrow{v^1|} \cdots .$$

◇

Of course, one could also consider bounded cohomology of groups with more general coefficients [Monod]. But in the following we will need only bounded cohomology with real coefficients.

Corollary (2.23). *Let G be a discrete group. Then any G-morphism between two strong relatively injective resolutions of the trivial bounded G-module $\mathbf{R}$ by bounded G-modules extending the identity $\mathbf{R} \longrightarrow \mathbf{R}$ induces the same isomorphism on the cohomology of the G-fixed point sets.*

In particular, the bounded cohomology $\widehat{H}^*(G)$ is well-defined (up to canonical isomorphisms).

Proof. This follows directly from Theorem (2.20). A detailed proof is given by Ivanov [Ivanov$_1$, Section (3.7)]. ☐

Each strong relatively injective resolution of the trivial G-module $\mathbf{R}$ gives rise to a seminorm on $\widehat{H}^*(G)$. The topology on $\widehat{H}^*(G)$ is independent of the chosen resolution, but the seminorm does depend on the resolution. However, it is known that the standard resolution (compare Example (2.19)) induces the minimum of all these seminorms [Ivanov$_1$, Theorem (3.6)], the so-called **canonical seminorm**.

As in the case of ordinary group cohomology, each group homomorphism $f\colon G \longrightarrow H$ induces a homomorphism $\widehat{H}^*(f)\colon \widehat{H}^*(H) \longrightarrow \widehat{H}^*(G)$ (for a rigorous definition see [Ivanov$_1$, Section (3.7)]). This turns bounded cohomology of groups into a functor.

As one might suspect, $\widehat{H}^*(G) \cong \widehat{H}^*(BG)$ if BG is a model of the classifying space of G. In fact, much more holds, as is explained in Section 2.4.

2.3.3 ■ Amenability

An interesting facet of bounded cohomology is that it ignores a large class of groups – the amenable groups. This gives rise to interesting applications to the simplicial volume (see Subsection 2.4.3).

Definition (2.24). A group G is called **amenable** if the there is a G-invariant function $m\colon B(G,\mathbf{R}) \longrightarrow \mathbf{R}$ such that for all $f \in B(G,\mathbf{R})$

$$\inf_{g\in G} f(g) \leq m(f) \leq \sup_{g\in G} f(g). \qquad \diamond$$

Actually, this might seem a far-fetched definition. But amenable groups turn out to be quite interesting – for example they are connected with the Banach-Tarski paradox [Runde, Chapter 0].

In the above definition, m should be interpreted as a kind of special mean on G. Obviously, all finite groups are amenable – in this case m can be taken as integration over G with respect to the counting measure and dividing by the number of elements in G. Furthermore, all Abelian groups are amenable [Runde, Example 1.1.5].

Ivanov shows that bounded cohomology cannot detect amenable groups [Ivanov$_1$, Section 3.8]:

Theorem (2.25). *Let A be a normal amenable subgroup of the discrete group G. Then the map $\widehat{H}^*(G/A) \longrightarrow \widehat{H}^*(G)$ induced by the canonical homomorphism $G \longrightarrow G/A$ is an isometric isomorphism.*

Using the right invariant "mean" on A Ivanov proves that the standard resolution of the quotient group G/A also is strong relatively injective resolution of $\mathbf{R}$ by bounded G-modules. Moreover, the induced canonical isomorphism turns out to be isometric.

The theorem shows that the free group $\mathbf{Z} * \mathbf{Z}$ cannot be amenable: It can be shown that $\widehat{H}^2(\mathbf{Z} * \mathbf{Z})$ is not even finitely generated. The "proof" of Brooks [Brooks] gives good intuition about this fact, but is not entirely correct – a correct version can be found in the arcticles of Grigorchuk and Mitsumatsu [Grigorchuk], [Mitsumatsu].

2.4 ■ The mapping theorem

Perhaps surprisingly, the rather small differences between the definitions of singular and bounded cohomology lead to significant differences in the behaviour of both theories. For instance, Gromov and Ivanov showed that the bounded cohomology of spaces only depends on the fundamental group (Theorem (2.27)). More precisely, if $f\colon X \longrightarrow Y$ is a continuous map, there is a commutative diagram

$$\begin{tikzcd} \widehat{H}^*(Y) \arrow[r, "\widehat{H}^*(f)"] \arrow[d, "\cong"'] & \widehat{H}^*(X) \arrow[d, "\cong"] \\ \widehat{H}^*(\pi_1(Y)) \arrow[r, "\widehat{H}^*(\pi_1(f))"'] & \widehat{H}^*(\pi_1(X)) \end{tikzcd}$$

whose vertical arrows are canonical isometric isomorphisms. This diagram allows us to understand the bounded cohomology of spaces via Ivanov's transparent axiomatic approach of Section 2.3.

2.4.1 ■ Bounded cohomology and the fundamental group

One proceeds in two steps to obtain the fact that bounded cohomology of a space only depends on its fundamental group:

Theorem (2.26). *Let X be a simply connected CW-complex with countably many cells. Then the bounded cohomology of X is canonically isometrically isomorphic to the bounded cohomology of the one point space.*

Theorem (2.27). *Let X be a connected countable CW-complex. Then $\widehat{H}^*(X)$ is canonically isometrically isomorphic to $\widehat{H}^*\big(\pi_1(X)\big)$.*

Actually, these theorems remain true if we drop the condition that the spaces must be countable cellular. This situation is treated in Gromov's original exposition [Gromov$_2$, Section 3.1].

Quite understandable proofs of the above theorems can be found in the paper of Ivanov [Ivanov$_1$]. We will sketch these proofs: in the following, we view $\widehat{C}^k(X)$ as the set of bounded functions $S_k(X) \longrightarrow \mathbf{R}$.

To prove the first theorem a contracting chain homotopy

$$\mathbf{R} \xleftarrow{K_0} \widehat{C}^0(X) \xleftarrow{K_1} \widehat{C}^1(X) \xleftarrow{K_2} \cdots$$

with $\|K_n\| \leq 1$ for all $n \in \mathbf{N}$ is constructed explicitly using "towers of connective covers." Here it is needed that X is a countable CW-complex to ensure that its homotopy groups are all countable [Lundell, Weingram, Theorem IV 6.1]. This implies that there is a model of the Eilenberg-Mac Lane space $K\big(\pi_n(X), n\big)$ which is a topological Abelian group G_n. Then there is such a tower

$$\cdots \xrightarrow{p_3} X_3 \xrightarrow{p_2} X_2 \xrightarrow{p_1} X_1 = X$$

consisting of principal G_n-bundles $p_n \colon X_{n+1} \longrightarrow X_n$. Ivanov shows that in this situation for each $n \in \mathbf{N}$ there is a chain map $A_n \colon \widehat{C}^*(X_{n+1}) \longrightarrow \widehat{C}^*(X_n)$ satisfying

$$A_n \circ \widehat{H}^*(p_n) = \mathrm{id} \quad \text{and} \quad \|A_n\| = 1.$$

Using these $(A_n)_{n\in\mathbf{N}}$, the contracting homotopy K_* can inductively be constructed.

In order to prove the second theorem, the first theorem is applied to show that (where $p\colon \widetilde{X} \longrightarrow X$ is the universal covering of X)

$$0 \longrightarrow \mathbf{R} \longrightarrow \widehat{C}^0(\widetilde{X}) \longrightarrow \widehat{C}^1(\widetilde{X}) \longrightarrow \cdots$$

is a strong relatively injective resolution of the trivial bounded $\pi_1(X)$-module $\mathbf{R}$. Here the left $\pi_1(X)$-action on $\widehat{C}^*(\widetilde{X})$ is given by the right $\pi_1(X)$-action on the universal covering space $\widetilde{X}$: for all $f \in \widehat{C}^k(X)$ and all $g \in \pi_1(X)$

$$g \cdot f := \big(\sigma \longmapsto f(x \mapsto \sigma(x) \cdot g)\big).$$

Thus the bounded cohomology of the discrete group $\pi_1(X)$ can be computed as the cohomology of the complex

$$0 \longrightarrow \widehat{C}^0(\widetilde{X})^{\pi_1(X)} \longrightarrow \widehat{C}^1(\widetilde{X})^{\pi_1(X)} \longrightarrow \cdots$$

The map $\widehat{C}^*(X) \longrightarrow \widehat{C}^*\big(\widetilde{X}\big)$ induced by p establishes an isomorphism between $\widehat{C}^*(X)$ and $\widehat{C}^*(\widetilde{X})^{\pi_1(X)}$ commuting with the differentials. Hence the bounded cohomology of $\pi_1(X)$ coincides with the cohomology of the complex

$$0 \longrightarrow \widehat{C}^0(X) \xrightarrow{\partial} \widehat{C}^1(X) \xrightarrow{\partial} \cdots$$

which is the bounded cohomology of the space X. A more detailed calculation reveals that the induced isomorphism is actually isometric.

Corollary (2.28). *If X is a connected countable CW-complex with amenable fundamental group, then $\widehat{H}^*(X) \cong \widehat{H}^*(\bullet)$.* $\square$

Example (2.29). The above relation between bounded cohomology and fundamental groups also yields that bounded cohomology cannot satisfy excision and that we cannot expect to find a cellular version of bounded cohomology: We have $\pi_1(S^1 \vee S^1) = \mathbf{Z} * \mathbf{Z}$, and $\widehat{H}^2(\mathbf{Z} * \mathbf{Z})$ is non-trivial (see Subsection 2.3.3), but $\widehat{H}^*(S^1) \cong \widehat{H}^*(\bullet)$. ■

Example (2.30). There is no interesting theory of characteristic classes for real vector bundles in bounded cohomology: For each $n \in \mathbf{N}$, the fundamental group of the classifying space $BO(n)$ is amenable. Therefore the bounded cohomology $\widehat{H}^*(BO(n))$ coincides with the bounded cohomology of the one point space.

However, there are interesting results about the supremum norm of the ordinary Euler class of orientable flat vector bundles [Gromov$_2$, page 231]. ■

2.4.2 ■ The mapping theorem

If $f\colon X \longrightarrow Y$ is a continuous map, chasing through all the definitions and isomorphisms shows that $\widehat{H}^*(f) = \widehat{H}^*(\pi_1(f))$ holds with respect to the isometric isomorphisms $\widehat{H}^*(X) \cong \widehat{H}^*(\pi_1(X))$ and $\widehat{H}^*(Y) \cong \widehat{H}^*(\pi_1(Y))$ of Theorem (2.27). Thus combining Theorem (2.27) and Theorem (2.25) yields:

Theorem (2.31) (Mapping Theorem for Bounded Cohomology). *Let X and Y be countable CW-complexes and let $f\colon X \longrightarrow Y$ be a continuous map. If the homomorphism $\pi_1(f)\colon \pi_1(X) \longrightarrow \pi_1(Y)$ is surjective and its kernel is amenable, then the induced homomorphism*

$$\widehat{H}^*(f) = \widehat{H}^*(\pi_1(f))\colon \widehat{H}^*(Y) \longrightarrow \widehat{H}^*(X)$$

is an isometric isomorphism.

By Corollary (2.11), in the situation of Theorem (2.31), the homomorphism $H_*(f,\mathbf{R})$ then also preserves the seminorm in homology.

Again, the mapping theorem remains true if we drop the condition for the spaces to be countable cellular [Gromov$_2$, Section 3.1].

2.4.3 ■ Applications to simplicial volume

In particular, we can apply the mapping theorem to obtain the following results about the simplicial volume:

Corollary (2.32). *Let M be an oriented closed connected manifold of positive dimension n, and let G be its fundamental group.*

1. *If $f\colon M \longrightarrow BG$ denotes the classifying map, then*
$$\|M\| = \|H_n(f,\mathbf{R})([M]_{\mathbf{R}})\|_1 .$$
2. *Moreover, if G is amenable, then $\|M\| = 0$.*

Proof. 1. By construction, $\pi_1(f)$ is an isomorphism. Hence the mapping theorem yields that $\widehat{H}^n(f)$ is an isometric isomorphism. Now Corollary (2.11) shows $\|M\| = \|H_n(f,\mathbf{R})([M]_{\mathbf{R}})\|_1$.

2. Corollary (2.28) implies that $\widehat{H}^n(M)$ vanishes. Hence
$$\|M\| = \left\|[M]_{\mathbf{R}}\right\|_1 = 0,$$
by duality (the first part of Theorem (2.7)). □

A nice argument from covering theory yields the following result [Lück$_2$, Exercise 14.11]:

Corollary (2.33). *Let M and N be oriented closed connected manifolds of the same dimension and let $f\colon M \longrightarrow N$ be a continuous map of degree 1. Then $\pi_1(f)$ is surjective. If the kernel of $\pi_1(f)$ is amenable, then*
$$\|M\| = \|N\| .$$

Example (2.34). However, the simplicial volume does not only depend on the fundamental group of the manifold: Let F_2 be the oriented closed connected surface of genus 2. Then F_2 and $F_2 \times S^2$ have the same fundamental group, but
$$\|F_2\| = 4 \quad \text{and} \quad \|F_2 \times S^2\| = 0$$
by Example (1.24) and Theorem (2.14) respectively. ■

Furthermore, one can derive from the mapping theorem that the simplicial volume is additive with respect to connected sums.

Theorem (2.35). *Let M and N be two oriented closed connected manifolds of the same dimension $n \geq 3$. Then*

$$\|M \# N\| = \|M\| + \|N\| .$$

Gromov's proof of this theorem combines beautiful geometric ideas of simplicial "tree-like" complexes with the mapping theorem (applied to the evident map $M \# N \longrightarrow M \vee N$). However, his exposition in "Volume and Bounded Cohomology" [Gromov$_2$, Section 3.5] is rather short.

2.5 ■ A special resolution

We close this chapter with the discussion of a special resolution which computes bounded cohomology. This resolution will be one of the ingredients needed to show that singular homology and measure homology are isometrically isomorphic (see Section 4.3).

Definition (2.36). Let X be an arcwise connected space with universal covering $\widetilde{X}$, and let $k \in \mathbf{N}$.

- Then $\pi_1(X)$ acts from the left on the vector space $\operatorname{map}(\widetilde{X}^{k+1}, \mathbf{R})$ of continuous functions $\widetilde{X}^{k+1} \longrightarrow \mathbf{R}$ via

  $$g \cdot f := \big((x_0, \dots, x_k) \mapsto f(x_0 \cdot g, \dots, x_k \cdot g)\big)$$

  for all $f \in \operatorname{map}(\widetilde{X}^{k+1}, \mathbf{R})$ and all $g \in \pi_1(X)$.

- The subset of bounded functions in $\operatorname{map}(\widetilde{X}^{k+1}, \mathbf{R})$ is denoted by $\mathsf{F}^*(X)$ and we use the abbreviation $\mathsf{I}^k(X)$ for the functions in $\mathsf{F}^k(X)$ which are invariant under the above $\pi_1(X)$-action. ◇

How can we turn $\mathsf{F}^*(X)$ into a cochain complex? The vector space $\widehat{C}^k(\widetilde{X})$ can be identified with the space of bounded functions $S_k(\widetilde{X}) \longrightarrow \mathbf{R}$ (under this identification, the norm $\|\cdot\|_\infty$ just becomes the supremum norm).

Now $\mathsf{F}^k(X)$ can be viewed as a subspace of $\widehat{C}^k(\widetilde{X})$, namely as the space of those bounded functions $S_k(\widetilde{X}) \longrightarrow \mathbf{R}$ which only depend on the vertices of the simplices (and are continuous in the vertices). In Gromov's terminology those functions would be called "straight bounded continuous cochains" [Gromov$_2$, Section 2.3].

It is clear that the coboundary operator on $\widehat{C}^*(\widetilde{X})$ restricts to $\mathsf{F}^*(X)$ and that the operations of $\pi_1(X)$ on $\mathsf{F}^*(X)$ and $\widehat{C}^*(\widetilde{X})$ are compatible with the above inclusion map. This makes $\mathsf{F}^*(X)$ a subcomplex of $\widehat{C}^*(\widetilde{X})$. In other words, the homomorphism

$$\begin{aligned} u^k\colon \mathsf{F}^k(X) &\longrightarrow \widehat{C}^k(\widetilde{X}) \\ f &\longmapsto \big(\sigma \mapsto f(\sigma(e_0),\dots,\sigma(e_k))\big) \end{aligned}$$

is a $\pi_1(X)$-equivariant cochain map.

Theorem (2.37). *Let X be a connected countable CW-complex and let $\widetilde{X}$ be its universal covering space.*

1. *Then the homomorphism*

$$\begin{aligned} \varepsilon\colon \mathbf{R} &\longrightarrow \mathsf{F}^0(X) = \operatorname{map}(\widetilde{X}, \mathbf{R}) \\ c &\longmapsto (x \mapsto c) \end{aligned}$$

   turns the cochain complex $\mathsf{F}^*(X)$ into a strong relatively injective resolution by bounded $\pi_1(X)$-modules of the trivial $\pi_1(X)$-module $\mathbf{R}$.

2. *Hence the cohomology of $\mathsf{I}^*(X)$ is canonically isomorphic to $\widehat{H}^*(\pi_1(X))$. Moreover, under this isomorphism the supremum norm on $\mathsf{F}^*(X)$ induces the canonical seminorm on $\widehat{H}^*(\pi_1(X))$.*

Corollary (2.38). *Let X be a connected countable CW-complex. Then $H^*\big(\mathsf{I}^*(X)\big)$ and $\widehat{H}^*(X)$ are isometrically isomorphic (by Theorem (2.27)).* □

A proof of the theorem is given in Monod's book [Monod, Theorem 7.4.5]. However, Monod does not explicitly write down a cochain map

$$\mathsf{I}^*(X) \longrightarrow \widehat{C}^*(X)$$

inducing this isometric isomorphism. Since we need a concrete description of this map in Section 4.3, we will now construct such a cochain map with the help of the uniqueness part of Theorem (2.20).

As mentioned in Section 2.4, the bounded cohomology groups $\widehat{H}^*(\pi_1(X))$ can be computed via the strong relatively injective resolution $\widehat{C}^*(\widetilde{X})$. Therefore we will compare the two complexes $\mathsf{I}^*(X)$ and $\widehat{C}^*(\widetilde{X})$: The "inclusion" map u^* from above fits into the ladder

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{R} \arrow[r, "\varepsilon"] \arrow[d, "\mathrm{id}"'] & \mathsf{F}^0(X) \arrow[r] \arrow[d, "u^0"'] & \mathsf{F}^1(X) \arrow[r] \arrow[d, "u^1"'] & \cdots \\
0 \arrow[r] & \mathbf{R} \arrow[r, "\varepsilon"'] & \widehat{C}^0(\widetilde{X}) \arrow[r] & \widehat{C}^1(\widetilde{X}) \arrow[r] & \cdots .
\end{tikzcd}$$

Now Corollary (2.23) implies that (after taking the $\pi_1(X)$-fixed points) u^* induces the canonical isomorphism

$$H^*\big(\mathsf{I}^*(X)\big) \longrightarrow H^*\big(\widehat{C}^*(\widetilde{X})^{\pi_1(X)}\big)$$

(which is isometric by Theorem (2.37)). As mentioned in Section 2.4, the isometric isomorphism

$$\widehat{p}^*\colon \widehat{H}^*(X) \longrightarrow H^*\big(\widehat{C}^*(\widetilde{X})^{\pi_1(X)}\big)$$

is induced by the universal covering map $p\colon \widetilde{X} \longrightarrow X$. Thus the desired isometric isomorphism $H^*\big(\mathsf{I}^*(X)\big) \longrightarrow \widehat{H}^*(X)$ is the homomorphism $H^*(w^*)$ making the triangle

$$\begin{tikzcd}
H^*\big(\mathsf{I}^*(X)\big) \arrow[rr, "H^*(w^*)"] \arrow[dr, "H^*(u|^*)"'] & & \widehat{H}^*(X) \arrow[dl, "\widehat{p}^*"] \\
& H^*\big(\widehat{C}^*(\widetilde{X})^{\pi_1(X)}\big) &
\end{tikzcd}$$

commutative (where $u|^*$ denotes the restriction of u^* to the $\pi_1(X)$-fixed points).

The inverse of $\widehat{p}^*$ on the level of cochains can be described as follows: for all $f \in \widehat{C}^k(\widetilde{X})^{\pi_1(X)}$ and $\sigma \in S_k(X)$ we define

$$\big(v^k(f)\big)(\sigma) := f(\widetilde{\sigma}),$$

where $\widetilde{\sigma}$ is some lift of σ with respect to p. This definition does not depend on the chosen lift since f is $\pi_1(X)$-invariant and any two lifts of σ only differ by multiplication with an element in $\pi_1(X)$ [Massey, Lemma V 8.1]. Since f is bounded, $v^k(f)$ is bounded on $S_k(X)$. Therefore linear extension yields $v^k(f) \in \widehat{C}^k(X)$.

Covering theory implies that for all $\sigma \in S_k(X)$ and all $j \in \{0, \dots, k\}$ each lift of the j-th face of σ is also the j-th face of some lift of σ. Hence

$$v^* \colon \widehat{C}^*(\widetilde{X})^{\pi_1(X)} \longrightarrow \widehat{C}^*(X)$$

is a cochain map. By construction, it is the inverse of the cochain map $\widehat{C}^*(X) \longrightarrow \widehat{C}^*(\widetilde{X})^{\pi_1(X)}$ induced by the universal covering map p.

Therefore, the composition

$$w^* := v^* \circ u|^* \colon \mathsf{I}^*(X) \longrightarrow \widehat{C}^*(X)$$

induces the canonical isometric isomorphism $H^*(\mathsf{I}^*(X)) \longrightarrow \widehat{H}^*(X)$. By definition, for all $f \in \mathsf{I}^k(X)$ and all $\sigma \in S_k(X)$ we obtain

$$\big(w^k(f)\big)(\sigma) = f\big(\widetilde{\sigma}(e_0), \dots, \widetilde{\sigma}(e_k)\big),$$

where $\widetilde{\sigma}$ is any lift of σ. This is the description we will use in Section 4.3.

3 Measure Homology

Measure homology is a generalisation of singular homology in the following way: singular chains with real coefficients are viewed as signed measures on the space of singular simplices whose mass is concentrated in finitely many points. In the measure homology chain complex also more complicated measures are allowed.

Measure homology was introduced in Thurston's lecture notes in 1979 [Thurston, page 6.6]. He already claimed that measure homology and singular homology should coincide. More extensive accounts are the papers of Zastrow [Zastrow] and Hansen [Hansen] and the book of Ratcliffe [Ratcliffe].

The motivation for the introduction of measure homology originates from the fact that measure homology can be used to calculate the simplicial volume, hence giving more room for constructions such as smearing (cf. Chapter 5). This smearing construction will be the key to the proof of the proportionality principle.

In Section 3.1, the technical background needed for the definition of measure homology is presented – in particular, a feasible topology on the set of smooth singular simplices will be introduced. In Section 3.2 measure homology is defined. The basic properties of measure homology are listed in Section 3.3, and there is also given a proof for the compatibility of measure homology with colimits. The next chapter is devoted to the proof that measure homology and singular homology are isometrically isomorphic.

3.1 ■ Prelude

In this section the basic objects involved in the definition of measure homology are examined. The first subsection introduces signed measures. In the later subsections, topologies for mapping spaces are provided and some technicalities concerning smooth maps on standard simplices are discussed.

3.1.1 ■ Signed measures

Signed measures are a slight generalisation of positive measures. As the name suggests, signed measures can also take negative values. The generalisation is quite straightforward; however, some additional care has to be taken when defining null sets or integrability with respect to a signed measure. The main advantage of signed measures is that the set of all finite signed measures on a given measurable space is a vector space over the real numbers.

Definition (3.1). Let $(X, \mathcal{A})$ be a measurable space.

- A map $\mu\colon \mathcal{A} \longrightarrow \mathbf{R} \cup \{\infty, -\infty\}$ is called a **signed measure** if $\mu(\emptyset) = 0$, not both ∞ and $-\infty$ are contained in the image of μ, and μ is σ-additive.
- A **null set** of a signed measure μ is a measurable set A with $\mu(B) = 0$ for all $B \in \mathcal{A}$ with $B \subset A$.
- A set $A \in \mathcal{A}$ is called **μ-positive** if $\mu(B) \geq 0$ for all $B \in \mathcal{A}$ with $B \subset A$. Analogously, **μ-negative** sets are defined.
- A **determination set of** μ is a subset D of X such that each measurable set contained in the complement of D is a μ-null set. $\diamond$

Of course, each ordinary positive measure is also a signed measure. If μ and ν are two positive measures, at least one of them being finite, then $\mu - \nu$ and $\nu - \mu$ are two signed measures. In fact, these are the *only* examples since each signed measure can be written as such a difference.

Theorem (3.2) (Hahn Decomposition). *Let μ be a signed measure on the measurable space $(X, \mathcal{A})$. Then there is a decomposition $X = P \cup N$ where P is a μ-positive set and N is a μ-negative set.*

This decomposition is essentially unique: if $X = P' \cup N'$ is another decomposition in a μ-positive and a μ-negative set, then the symmetric difference of P and P' (which equals the symmetric difference of N and N') is a μ-null set.

A proof is, for example, given in Elstrodt's textbook on measure theory and integration [Elstrodt, Satz 1.8 in Kapitel VII].

Definition (3.3). Let μ be a signed measure on $(X, \mathcal{A})$ and $X = P \cup N$ its Hahn decomposition.

- The (positive) measure

$$\begin{aligned} \mu^+ \colon \mathcal{A} &\longrightarrow \mathbf{R}_{\geq 0} \cup \{\infty\} \\ A &\longmapsto \mu(A \cap P) \end{aligned}$$

  is called **positive variation of** μ. Analogously, the restriction μ^- of $-\mu$ to the set N is called **negative variation of** μ. The decomposition $\mu = \mu^+ - \mu^-$ is the **Jordan decomposition of** μ.

- The **variation of** μ is defined as the sum

$$|\mu| := \mu^+ + \mu^-$$

  and the **total variation of** μ is given by

$$\|\mu\| := |\mu|(X) \in \mathbf{R}_{\geq 0} \cup \{\infty\}. \qquad \diamondsuit$$

Since the Hahn decomposition is unique up to null sets, the positive and the negative variation of a signed measure are indeed well-defined. Moreover, the sum $\mu^+ + \mu^-$ makes sense because at least one of these two measures must be finite.

Sometimes it is useful to have another description of the total variation:

Lemma (3.4). *Let $(X, \mathcal{A})$ be a measurable space and μ a signed measure on $(X, \mathcal{A})$. Then*

$$\|\mu\| = \sup_{A \in \mathcal{A}} \mu(A) - \inf_{B \in \mathcal{A}} \mu(B).$$

Proof. Let $X = P \cup N$ be the Hahn decomposition of μ. By definition,

$$\|\mu\| = |\mu|(X) = \mu(P) - \mu(N).$$

This clearly shows "$\leq$". On the other hand, for all $A, B \in \mathcal{A}$

$$\begin{aligned}\mu(A) - \mu(B) &= \mu^+(A) - \mu^-(A) - \mu^+(B) + \mu^-(B) \\ &\leq \mu^+(A) + \mu^-(B) \\ &\leq \mu^+(X) + \mu^-(X) \\ &= |\mu|(X) \\ &= \|\mu\|\end{aligned}$$

since μ^+ and μ^- are positive measures. □

Definition (3.5). Let X be a topological space. The **Borel σ-algebra** is the σ-algebra on X generated by all open sets of X. ◇

Definition (3.6). Let $(X, \mathcal{A})$ be a measurable space and $f\colon (X, \mathcal{A}) \longrightarrow \mathbf{R}$ a measurable function (with respect to the Borel σ-algebra on $\mathbf{R}$). If μ is a signed measure on $(X, \mathcal{A})$, we call f **integrable with respect to** μ if f is integrable with respect to the positive measures μ^+ and μ^- in the usual sense. The **integral of f with respect to** μ is then defined as

$$\int f\, d\mu := \int f\, d\mu^+ - \int f\, d\mu^-.$$ ◇

Lemma (3.7). *Let X be a topological space. The set of all signed measures on the Borel σ-algebra of X possessing a compact determination set and finite total variation is a real vector space. The total variation turns this vector space into a normed vector space.*

Proof. Let μ and ν be two such measures and let $\alpha \in \mathbf{R}$. Obviously, $\alpha \cdot \mu$ again is a signed measure. Each determination set of μ is a determination set of $\alpha \cdot \mu$ and

$$\|\alpha \cdot \mu\| = |\alpha| \cdot \|\mu\|.$$

Since both μ and ν have finite total variation, their value on each Borel set is finite. Hence the sum $\mu + \nu$ is well-defined and again a signed measure. Furthermore, if D_μ and D_ν are (compact) determination sets of μ and ν respectively then $D_\mu \cup D_\nu$ is a (compact) determination set of $\mu + \nu$.

Using Lemma (3.4) we get (where $\mathcal{A}$ is the Borel σ-algebra on X)

$$\begin{aligned}\|\mu + \nu\| &= \sup_{A\in\mathcal{A}}\big(\mu(A) + \nu(A)\big) - \inf_{B\in\mathcal{A}}\big(\mu(B) + \nu(B)\big)\\ &\leq \sup_{A\in\mathcal{A}} \mu(A) + \sup_{A\in\mathcal{A}} \nu(A) - \inf_{B\in\mathcal{A}} \mu(B) - \inf_{B\in\mathcal{A}} \nu(B)\\ &= \|\mu\| + \|\nu\|\,.\end{aligned}$$

In particular, $\|\mu + \nu\|$ is finite. $\square$

Definition (3.8). Let $f\colon X \longrightarrow Y$ be a **Borel function**, i.e., measurable with respect to the Borel σ-algebra. Moreover, let μ be a signed measure on the Borel σ-algebra of X. Then μ induces a signed measure μ^f on Y given by

$$\mu^f(A) := \mu\big(f^{-1}(A)\big)$$

for all Borel subsets $A \subset Y$. $\diamond$

Lemma (3.9). *Let $f\colon X \longrightarrow Y$ be a continuous function and let μ be a signed measure on the Borel σ-algebra of X possessing a compact determination set and finite total variation. Then the induced measure μ^f also has a compact determination set and finite total variation.*

Proof. It is easy to see that continuous functions are Borel functions since the Borel σ-algebra is generated by the open sets. Hence μ^f is well-defined.

If D is a compact determination set of μ, then the image $f(D)$ of course is a determination set of μ^f. Since f is continuous, $f(D)$ is compact.

Writing $\mathcal{A}_X$ and $\mathcal{A}_Y$ for the Borel σ-algebras on X and Y respectively we compute the total variation of μ^f via Lemma (3.4) as follows

$$\begin{aligned}\|\mu^f\| &= \sup_{A\in\mathcal{A}_Y} \mu^f(A) - \inf_{B\in\mathcal{A}_Y} \mu^f(B)\\ &= \sup_{A\in\mathcal{A}_Y} \mu\big(f^{-1}(A)\big) - \inf_{B\in\mathcal{A}_Y} \mu\big(f^{-1}(B)\big)\\ &\leq \sup_{A\in\mathcal{A}_X} \mu(A) - \inf_{B\in\mathcal{A}_X} \mu(B)\\ &= \|\mu\|\,.\end{aligned}$$

Hence μ^f also has finite total variation. $\square$

3.1.2 ■ Mapping spaces

Since we want to define measure chains as signed measures on some set of singular simplices, we have to introduce an appropriate topology on the corresponding mapping space. Unfortunately, the compact open topology is too coarse for our purposes. So we need another topology – the so-called $\mathcal{C}^1$-topology defined on the space of smooth maps of smooth manifolds –, which takes into account the smooth structure.

Definition (3.10). Let X and Y be two topological spaces and $\operatorname{map}(X, Y)$ the set of continuous maps $X \longrightarrow Y$.

- If $K \subset X$ is compact and $U \subset Y$ is open, we write
$$U^K := \{f \in \operatorname{map}(X, Y) \mid f(K) \subset U\}.$$
- The **compact open topology** on the mapping space $\operatorname{map}(X, Y)$ is the topology with the subbase
$$\{U^K \mid K \subset X \text{ compact}, U \subset Y \text{ open}\}. \qquad \diamond$$

Remark (3.11). In particular, for all $x \in X$ the evaluation map
$$\begin{aligned} \operatorname{map}(X, Y) &\longrightarrow Y \\ f &\longmapsto f(x) \end{aligned}$$
is continuous. □

Definition (3.12). Let M and N be two smooth manifolds without boundary. As sets $\operatorname{map}_\infty(M, N)$ and $\operatorname{map}_\infty^{\mathrm{co}}(M, N)$ are just the sets of smooth maps $M \longrightarrow N$.

- The topology on $\operatorname{map}_\infty^{\mathrm{co}}(M, N) \subset \operatorname{map}(M, N)$ is the subspace topology given by the compact open topology on $\operatorname{map}(M, N)$.
- The topology on $\operatorname{map}_\infty(M, N)$, called **$\mathcal{C}^1$-topology**, is the topology consisting of all inverse images of the form $T^{-1}(U)$ where U is an open subset of $\operatorname{map}(TM, TN)$ and T is the differential:
$$\begin{aligned} T\colon \operatorname{map}_\infty(M, N) &\longrightarrow \operatorname{map}(TM, TN) \\ f &\longmapsto Tf. \end{aligned}$$
I.e., the injective map $T\colon \operatorname{map}_\infty(M, N) \longrightarrow \operatorname{map}(TM, TN)$ is turned into a homeomorphism onto its image $T\bigl(\operatorname{map}_\infty(M, N)\bigr)$. $\diamond$

More information about the $\mathcal{C}^1$-topology and similar topologies can be found in Hirsch's book [Hirsch$_1$, Chapter 2]. The above description is taken from a paper of Hirsch [Hirsch$_2$, page 244].

Remark (3.13). The compact open topology is the same as the topology of uniform convergence on compact subsets [Dugundji, Theorem XII 7.2]. Similarly, the $\mathcal{C}^1$-topology is the topology of uniform convergence on compact subsets not only of the function itself but also of its differential. $\diamond$

Lemma (3.14). *Let M and N be two smooth manifolds without boundary. Then the $\mathcal{C}^1$-topology on $\operatorname{map}_\infty(M,N)$ is finer than the compact open topology.*

Proof. Denoting the zero section $M \hookrightarrow TM$ by i and the bundle projection $TN \longrightarrow N$ by π, we get for all compact sets $K \subset M$ and all open sets $U \subset N$ that

$$\begin{aligned}
&T\big(U^K \cap \operatorname{map}_\infty(M,N)\big)\\
&\quad = \big\{Tf\colon TM \longrightarrow TN \;\big|\; f \in U^K \cap \operatorname{map}_\infty(M,N)\big\}\\
&\quad = \big\{Tf\colon TM \longrightarrow TN \;\big|\; f \in \operatorname{map}_\infty(M,N), Tf(i(K)) \subset \pi^{-1}(U)\big\}\\
&\quad = \big\{g \in \operatorname{map}(TM,TN) \;\big|\; g(i(K)) \subset \pi^{-1}(U)\big\} \cap T\big(\operatorname{map}_\infty(M,N)\big).
\end{aligned}$$

Since i and π are continuous, $i(K) \subset TM$ is compact and $\pi^{-1}(U) \subset TN$ is open. Thus the set $U^K \cap \operatorname{map}_\infty(M,N)$ is open in $\operatorname{map}_\infty(M,N)$. $\square$

Lemma (3.15). 1. *Let X, Y, and Z be topological spaces and $f \in \operatorname{map}(X,Y)$, $g \in \operatorname{map}(Y,Z)$. Then the maps*

$$\begin{aligned}
\operatorname{map}(Y,Z) &\longrightarrow \operatorname{map}(X,Z)\\
h &\longmapsto h \circ f
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{map}(X,Y) &\longrightarrow \operatorname{map}(X,Z)\\
h &\longmapsto g \circ h
\end{aligned}$$

are continuous.

2. *Let L, M, N be smooth manifolds without boundary and $f \in \operatorname{map}_\infty(L,M)$, $g \in \operatorname{map}_\infty(M,N)$. Then the maps*

$$\begin{aligned}
\operatorname{map}_\infty(M,N) &\longrightarrow \operatorname{map}_\infty(L,N)\\
h &\longmapsto h \circ f
\end{aligned}$$

and

$$\begin{aligned} \mathrm{map}_\infty(L, M) &\longrightarrow \mathrm{map}_\infty(L, N) \\ h &\longmapsto g \circ h \end{aligned}$$

are continuous.

Proof. 1. This can easily be seen using the definition of the compact open topology.

2. Applying the first part to $Tf \in \mathrm{map}(TL, TM)$ and $Tg \in \mathrm{map}(TM, TN)$ proves the second part. $\square$

3.1.3 ■ "Smooth" maps on standard simplices

Since the standard simplex Δ^k is quite angular, some care has to be taken when defining the set of all smooth maps $\Delta^k \longrightarrow M$, where M is some smooth manifold without boundary. The elements of $\mathrm{map}_\infty(\Delta^k, M)$ should be thought of as being smooth maps on some slightly larger set without "corners." More precisely:

Definition (3.16). Let M be a smooth manifold without boundary and $k \in \mathbf{N}$.

- The k-dimensional hyperplane in $\mathbf{R}^{k+1}$ spanned by the vertices of the standard k-simplex Δ^k will be denoted by $\square^k$. We then write Λ^k for the set of all open sets $V \subset \square^k$ (in the subspace topology) containing Δ^k.
- A map $f\colon \Delta^k \longrightarrow M$ is **smooth** if there is a $V \in \Lambda^k$ and a smooth map $F \in \mathrm{map}_\infty(V, M)$ such that

  $$F|_{\Delta^k} = f.$$

  The pair (F, V) is called an **extension of** f.
- A map $f\colon M \longrightarrow \Delta^k$ is called **smooth** if the composition $i \circ f$ is smooth, where i is an affine isomorphism $\square^k \longrightarrow \mathbf{R}^k$ composed with the inclusion $\Delta^k \hookrightarrow \square^k$. We also use this terminology if M happens to be a standard simplex and f is smooth in the above sense.
- We write $\mathrm{map}_\infty(\Delta^k, M)$ and $\mathrm{map}_\infty(M, \Delta^k)$ for the set of all smooth maps $\Delta^k \longrightarrow M$ and the set of all smooth maps $M \longrightarrow \Delta^k$, respectively. $\diamond$

Remark (3.17). If M is a smooth manifold without boundary and $k \in \mathbf{N}$ then we obviously get an inclusion $\mathrm{map}_\infty(\Delta^k, M) \hookrightarrow S_k(M)$. $\diamond$

The mapping space $\mathrm{map}(\Delta^k, M)$ can be topologised as follows:

Definition (3.18). Let M be a smooth manifold without boundary or a standard simplex, and let $k \in \mathbf{N}$. We use the notation $\mathrm{map}_\infty^{\mathrm{co}}(\Delta^k, M)$ for the set $\mathrm{map}_\infty(\Delta^k, M) \subset \mathrm{map}(\Delta^k, M)$ endowed with the subspace topology. $\diamond$

However, we will also need some kind of $\mathcal{C}^1$-topology on $\mathrm{map}_\infty(\Delta^k, M)$. So we introduce the differential of a smooth map $\Delta^k \longrightarrow M$:

Definition (3.19). For $k \in \mathbf{N}$ we use the notation

$$T\Delta^k := \Delta^k \times \mathbf{R}^k.$$

Moreover, if $V \in \Lambda_k$, we will always presuppose the canonical identification

$$TV = V \times \mathbf{R}^k$$

of the tangent bundle of the smooth k-manifold V. $\diamond$

Definition (3.20). Let M be a smooth manifold without boundary or a standard simplex, and let $k \in \mathbf{N}$. For a smooth map $f \in \mathrm{map}_\infty(\Delta^k, M)$ we define its **differential** by

$$Tf := TF|_{T\Delta^k} \colon T\Delta^k \longrightarrow TM$$

where (F, V) is some extension of f. $\diamond$

This differential is indeed well-defined:

Lemma (3.21). *Let M be a smooth manifold, $k \in \mathbf{N}$, and $f \in \mathrm{map}_\infty(\Delta^k, M)$. Suppose (F_1, V_1) and (F_2, V_2) are two extensions of f. Then*

$$TF_1|_{\pi_1^{-1}(\Delta^k)} = TF_2|_{\pi_2^{-1}(\Delta^k)}$$

where $\pi_1 \colon TV_1 \longrightarrow V_1$ and $\pi_2 \colon TV_2 \longrightarrow V_2$ are the bundle projections.

Proof. Since F_1 and F_2 are smooth, their differentials TF_1 and TF_2 are continuous. Furthermore,

$$TF_1|_{\pi_1^{-1}((\Delta^k)^\circ)} = TF_2|_{\pi_2^{-1}((\Delta^k)^\circ)}$$

since F_1 and F_2 extend f. For each $(x, v) \in \pi_1^{-1}(\Delta^k) = \pi_2^{-1}(\Delta^k)$ there is a sequence $(x_n)_{n \in \mathbf{N}} \subset (\Delta^k)^\circ$ such that

$$x = \lim_{n \to \infty} x_n,$$

which implies $\lim_{n \to \infty}(x_n, v) = (x, v)$ in $\pi_1^{-1}(\Delta^k) = \pi_2^{-1}(\Delta^k)$. Hence we conclude

$$\begin{aligned} TF_1(x,v) &= TF_1\big(\lim_{n\to\infty}(x_n,v)\big) = \lim_{n\to\infty} TF_1(x_n,v) \\ &= \lim_{n\to\infty} TF_2(x_n,v) \quad = TF_2\big(\lim_{n\to\infty}(x_n,v)\big) \\ &= TF_2(x,v). \end{aligned}$$

□

Having defined the differential of a smooth map $\Delta^k \longrightarrow M$, we are now able to introduce the $\mathcal{C}^1$-topology on $\operatorname{map}_\infty(\Delta^k, M)$:

Definition (3.22). Let M be a smooth manifold without boundary or a standard simplex, and let $k \in \mathbf{N}$. The **$\mathcal{C}^1$-topology on** $\operatorname{map}_\infty(\Delta^k, M)$ is the topology induced by the compact open topology on $\operatorname{map}(T\Delta^k, TM)$ pulled back by the differential (as defined in Definition (3.20))

$$\begin{aligned} T\colon \operatorname{map}_\infty\big(\Delta^k, M\big) &\longrightarrow \operatorname{map}\big(T\Delta^k, TM\big) \\ f &\longmapsto Tf. \end{aligned}$$

◇

Lemma (3.23). *Let M be a smooth manifold without boundary or a standard simplex, and let $k \in \mathbf{N}$. The topology of $\operatorname{map}_\infty(\Delta^k, M)$ is finer than the topology of $\operatorname{map}^{\mathrm{co}}_\infty(\Delta^k, M)$.*

Proof. We can use literally the same proof as in Lemma (3.14). □

Remark (3.24). In the same way we can derive statements corresponding to the second part of Lemma (3.15). ◇

The following two rather technical results are needed for the proof that measure homology is compatible with colimits.

Lemma (3.25). *Let M be a smooth manifold without boundary and $N \subset M$ an open subset. Let $k \in \mathbf{N}$.*

1. *Then the map*
$$j\colon \operatorname{map}_\infty\big(\Delta^k, N\big) \longrightarrow \operatorname{map}_\infty\big(\Delta^k, M\big),$$
induced by the inclusion $N \hookrightarrow M$, is continuous, injective, and open.

2. *Let* $\operatorname{im} j := j\big(\mathrm{map}_\infty(\Delta^k, N)\big) \subset \mathrm{map}_\infty(\Delta^k, M)$ *(endowed with the subspace topology). Then the inverse* $\operatorname{im} j \longrightarrow \mathrm{map}_\infty(\Delta^k, N)$ *of* j *is continuous.*

Proof. 1. Since the inclusion $N \hookrightarrow M$ is smooth, continuity of j follows from Remark (3.24). The fact that j is injective is clear because N is Borel in M.

The map j is open: Namely, since j is injective, taking the image of j is compatible with intersections. Hence it suffices to show that j maps sets of the form $T^{-1}(U^K)$ with open $U \subset TN$ and compact $K \subset T\Delta^k$ to open sets in $\mathrm{map}_\infty(\Delta^k, M)$. By definition,

$$\begin{aligned}
&j\big(T^{-1}(U^K)\big) \\
&\quad = j\big(\{f \in \mathrm{map}_\infty(\Delta^k, N) \mid Tf(K) \subset U\}\big) \\
&\quad = \big\{i \circ f \mid f \in \mathrm{map}_\infty(\Delta^k, N), Tf(K) \subset U\big\} \\
&\quad = \big\{f \in \mathrm{map}_\infty(\Delta^k, M) \mid Tf(K) \subset U\big\} \cap \big\{f \in \mathrm{map}(\Delta^k, M) \mid f(\Delta^k) \subset N\big\}
\end{aligned}$$

where $i\colon N \hookrightarrow M$ is the inclusion. Since N is open in M, the set U must be open in TM. Thus the first set is open in $\mathrm{map}_\infty(\Delta^k, M)$. Lemma (3.23) shows that the second set is also open in $\mathrm{map}_\infty(\Delta^k, M)$. Thus j is open.

2. This is an immediate consequence of the first part. □

Corollary (3.26). *Let* M *be a smooth manifold without boundary and let* $(U_k)_{k\in\mathbf{N}}$ *be an ascending family of open sets covering* M*. Let* $j \in \mathbf{N}$*. For each* $k \in \mathbf{N}$ *the map* $\mathrm{map}_\infty(\Delta^j, U_k) \longrightarrow \mathrm{map}_\infty(\Delta^j, M)$ *induced by the inclusion* $U_k \hookrightarrow M$ *is denoted by* j_k*. Then* $\big(j_k(\mathrm{map}_\infty(\Delta^j, U_k))\big)_{k\in\mathbf{N}}$ *is an ascending open covering of* $\mathrm{map}_\infty(\Delta^j, M)$*.*

Proof. It is clear that this family is ascending. By Lemma (3.25) all the images $j_k\big(\mathrm{map}_\infty(\Delta^j, U_k)\big)$ are open in $\mathrm{map}_\infty(\Delta^j, M)$.

Moreover, $\mathrm{map}_\infty(\Delta^j, M)$ is covered by $\big(j_k(\mathrm{map}_\infty(\Delta^j, U_k))\big)_{k\in\mathbf{N}}$: namely, if $f \in \mathrm{map}_\infty(\Delta^j, M)$, then the image $f(\Delta^j) \subset M$ is compact. Since $(U_k)_{k\in\mathbf{N}}$ is an ascending open covering of M, there is some $k \in \mathbf{N}$ such that $f(\Delta^j) \subset U_k$. If (F, V) is an extension of f, then $V' := F^{-1}(U_k)$ is an open subset of V containing Δ^j, and the pair $\big(F|_{V'}, V'\big)$ shows that there is an $\overline{f} \in \mathrm{map}_\infty(\Delta^j, U_k)$ satisfying

$$f = j_k(\overline{f}).$$

□

3.2 ■ Definition of measure homology

Measure homology is a generalisation of singular homology in the following sense: Let X be a topological space, $k \in \mathbf{N}$, and let $S \subset \operatorname{map}(\Delta^k, X)$ be some set of simplices. The idea of measure homology is to think of a singular chain $\sum_{\sigma \in S} a_\sigma \cdot \sigma$ with real coefficients as a signed measure on S having the mass a_σ on the set $\{\sigma\}$. The measure chain complex will consist of *all* signed measures on S (modulo some finiteness condition). In particular, certain "infinite" singular chains are allowed. Thus the measure chain complex is larger than the singular chain complex and hence gives more room for certain constructions such as smearing (see Section 5.4). The other side of the coin is that it is quite hard to get a geometric intuition of more complicated measure chains.

Depending on the choice of the mapping space S (and its topology) there are two main flavours of measure homology:

- One for general (pairs of) spaces using the compact open topology on the set of all singular simplices, and
- one for (pairs of) smooth manifolds using the $\mathcal{C}^1$-topology on the set of smooth singular simplices.

However, our application of measure homology in Chapter 5 compels us to use the smooth version. So we will only introduce this version in detail.

Measure homology was introduced by Thurston [Thurston, page 6.6]. Some basic properties of smooth measure homology are provided in Ratcliffe's book [Ratcliffe, §11.5]. A thorough treatment of measure homology for general spaces is given by Zastrow and Hansen [Zastrow], [Hansen]. In Theorem 3.4 of Zastrow's paper a way is described how to derive the corresponding results and proofs for the smooth theory. Just as Zastrow, we refer to those adapted statements of his paper by numbers such as, e.g., 4.5^{diff} instead of 4.5. Unfortunately, it is probably more transparent to provide a new proof for the compatibility with colimits instead of adapting Hansen's result [Hansen, Proposition 5.1]. Therefore we give a full proof at the end of this chapter.

Definition (3.27). Let M be a smooth manifold without boundary and $k \in \mathbf{N}$.

- The k**-th measure chain group**, denoted by $\mathcal{C}_k(M)$, is the $\mathbf{R}$-vector space of all signed measures on the Borel σ-algebra of the mapping space $\mathrm{map}_\infty(\Delta^k, M)$ possessing a compact determination set and finite total variation. Its elements are called **measure** k**-chains**.
- For each $j \in \{0, \dots, k+1\}$ the inclusion $\partial_j \colon \Delta^k \longrightarrow \Delta^{k+1}$ of the j-th face induces by Remark (3.24) and Lemma (3.9) a homomorphism (which we will also denote by ∂_j)

$$\begin{aligned} \partial_j \colon \mathcal{C}_{k+1}(M) &\longrightarrow \mathcal{C}_k(M) \\ \mu &\longmapsto \mu^{(\sigma \mapsto \sigma \circ \partial_j)} \end{aligned}$$

We then define the **boundary operator** of measure chains by

$$\begin{aligned} \partial \colon \mathcal{C}_{k+1}(M) &\longrightarrow \mathcal{C}_k(M) \\ \mu &\longmapsto \sum_{j=0}^{k+1} (-1)^j \cdot \partial_j(\mu). \end{aligned} \qquad \diamond$$

Without the condition on the variation, we would not get a vector space (since signed measures cannot be added in general). The compact determination sets play in some way the rôle of the compactness of the standard simplex (as can be seen in Lemma (3.45)) and the finiteness of singular chains in singular homology.

Just as in singular homology theory, we suppress the dependency of ∂ on k in the notation.

Lemma (3.28). *Let M be a smooth manifold without boundary and let $k \in \mathbf{N}$. Then $\big(\mathcal{C}_*(M), \partial\big)$ is a chain complex.*

Proof. This is shown in [Zastrow, Corollary 2.9^{diff}] and [Ratcliffe, Lemma 1 in §11.5]. $\square$

Statements like the previous lemma are proved by Zastrow, using a more general framework which permits to translate some properties of singular homology to the setting of measure homology [Zastrow, Theorem 2.1^{diff}].

Due to the above lemma, we can naturally define a homology theory:

Definition (3.29). Let M be a smooth manifold without boundary and $k \in \mathbf{N}$. The $\mathbf{R}$-vector space

$$\mathcal{H}_k(M) := \frac{\ker\bigl(\partial\colon \mathcal{C}_k(M) \longrightarrow \mathcal{C}_{k-1}(M)\bigr)}{\operatorname{im}\bigl(\partial\colon \mathcal{C}_{k+1}(M) \longrightarrow \mathcal{C}_k(M)\bigr)}$$

is called the **k-th measure homology group of** M. $\diamond$

In view of the goal to compute the simplicial volume via measure homology, we introduce a norm on measure homology using the total variation:

Definition (3.30). Let M be a smooth manifold without boundary and let $k \in \mathbf{N}$. The total variation $\|\cdot\|$ induces a seminorm on $\mathcal{H}_k(M)$ as follows: For all $\mu \in \mathcal{C}_k(M)$ we define

$$\|\mu\|_{\mathrm{mh}} := \inf\bigl\{\|\nu\| \bigm| \nu \in \mathcal{C}_k(M), \partial(\nu) = 0, [\nu] = \mu\bigr\}. \qquad \diamond$$

Just like in singular homology, there is also a notion of relative homology groups:

Definition (3.31). Let M be a smooth manifold without boundary and let N be a smooth submanifold without boundary. Due to Remark (3.24) and Lemma (3.9), the inclusion $N \hookrightarrow M$ induces a homomorphism $\mathcal{C}_*(N) \longrightarrow \mathcal{C}_*(M)$, which is injective since N is measurable. Hence for $k \in \mathbf{N}$ we can define the **k-th relative measure chain group** as the quotient vector space

$$\mathcal{C}_k(M,N) := \frac{\mathcal{C}_k(M)}{\mathcal{C}_k(N)}. \qquad \diamond$$

The boundary operator on the absolute measure chain groups clearly induces a boundary operator on the relative measure chain groups, turning them into a chain complex.

Definition (3.32). Let M be a smooth manifold without boundary and let $N \subset M$ be a smooth submanifold without boundary. For each $k \in \mathbf{N}$ the **k-th relative measure homology group** is defined to be the quotient

$$\mathcal{H}_k(M,N) := \frac{\ker\bigl(\partial\colon \mathcal{C}_k(M,N) \longrightarrow \mathcal{C}_{k-1}(M,N)\bigr)}{\operatorname{im}\bigl(\partial\colon \mathcal{C}_{k+1}(M,N) \longrightarrow \mathcal{C}_k(M,N)\bigr)}. \qquad \diamond$$

Remark (3.33). Let M be a smooth manifold without boundary. Since there is only one measure on the empty set (the zero-measure), we get $\mathcal{C}_*(\emptyset) = 0$. This yields canonical isomorphisms

$$\begin{aligned}\mathcal{C}_*(M) &\cong \mathcal{C}_*(M,\emptyset),\\ \mathcal{H}_*(M) &\cong \mathcal{H}_*(M,\emptyset).\end{aligned} \qquad \diamond$$

3.3 ■ Basic properties of measure homology

In the category of smooth manifolds without boundary and smooth maps measure homology behaves like singular homology with real coefficients; actually, most of the proofs are based on the same ideas as the corresponding proofs for singular homology theory.

Lemma (3.34) (Dimension Axiom). *Measure homology satisfies the dimension axiom: For all $n \in \mathbf{N}$*

$$\mathcal{H}_n(\bullet) \cong \begin{cases} \mathbf{R} & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. It is easy to see that the chain complexes $(\mathcal{C}_*(\bullet), \partial)$ and $(C_*(\bullet, \mathbf{R}), \partial)$ are the same because for each $k \in \mathbf{N}$ there is just one map $\Delta^k \longrightarrow \bullet$ (which is smooth). Thus

$$\mathcal{H}_*(\bullet) \cong H_*(\bullet, \mathbf{R})$$

which proves the lemma. □

Lemma (3.35) (Naturality). *Each smooth map $f\colon M \longrightarrow N$ between smooth manifolds without boundary induces a homomorphism*

$$\mathcal{H}_*(f)\colon \mathcal{H}_*(M) \longrightarrow \mathcal{H}_*(N)$$

in measure homology. This turns $\mathcal{H}_*$ into a functor.

Proof. Since f is smooth, we get for each $k \in \mathbf{N}$ an induced map

$$\begin{aligned} \operatorname{map}_\infty(\Delta^k, M) &\longrightarrow \operatorname{map}_\infty(\Delta^k, N) \\ \sigma &\longmapsto f \circ \sigma, \end{aligned}$$

which is continuous by Remark (3.24). Hence Lemma (3.9) shows that we obtain an induced map

$$\begin{aligned} \mathcal{C}_k(M) &\longrightarrow \mathcal{C}_k(N) \\ \mu &\longmapsto \mu^{(\sigma \mapsto f \circ \sigma)} \end{aligned}$$

which obviously is a homomorphism. This yields a chain map $\mathcal{C}_*(M) \longrightarrow \mathcal{C}_*(N)$ [Zastrow, Lemma 2.10^{diff}]. Hence we get an induced homomorphism $\mathcal{H}_*(M) \longrightarrow \mathcal{H}_*(N)$. It is clear that this assignment is functorial. □

Theorem (3.36) (Long Exact Pair Sequence). *There is an exact pair sequence for measure homology, i.e., if M is a smooth manifold without boundary and $N \subset M$ is a smooth submanifold without boundary, then the sequence*

$$\cdots \longrightarrow \mathcal{H}_{k+1}(M,N) \longrightarrow \mathcal{H}_k(N) \xrightarrow{\mathcal{H}_k(i)} \mathcal{H}_k(M) \xrightarrow{\mathcal{H}_k(j)} \mathcal{H}_k(M,N) \longrightarrow \cdots$$

*is exact for all $k \in \mathbf{N}$. Here $i\colon N \hookrightarrow M$ and $j\colon (M,\emptyset) \hookrightarrow (M,N)$ are the inclusions. The so-called **connecting homomorphism** $\mathcal{H}_{*+1}(M,N) \longrightarrow \mathcal{H}_*(N)$ is natural in M and N.*

Proof. The definition of relative measure chain groups yields a short exact sequence

$$0 \longrightarrow \mathcal{C}_k(N) \longrightarrow \mathcal{C}_k(M) \longrightarrow \mathcal{C}_k(M,N) \longrightarrow 0\,,$$

where the maps are induced by the (smooth) inclusions i and j. Therefore we obtain the desired long exact homology sequence [Weibel, Theorem 1.31]. □

Theorem (3.37) (Homotopy Invariance). *Measure homology is a homotopy invariant functor, i.e., if $f, g\colon M \longrightarrow N$ are homotopic smooth maps of smooth manifolds without boundary, then*

$$\mathcal{H}_*(f) = \mathcal{H}_*(g)\colon \mathcal{H}_*(M) \longrightarrow \mathcal{H}_*(N).$$

A proof is given in [Zastrow, 5.3^{diff}].

Remark (3.38). One might suspect that it is necessary for f and g in the above theorem to be *smoothly* homotopic. But smooth homotopic maps are always smoothly homotopic [Lee_2, Proposition 10.22]. ◇

Using a barycentric subdivision process, Zastrow proves the following theorem [Zastrow, Theorem 4.1^{diff}]:

Theorem (3.39) (Excision). *Measure homology satisfies the following variation of the excision axiom: Let M be a smooth manifold without boundary and let $W \subset M$ be a smooth submanifold without boundary. If $U \subset W$ is a subset such that $M \setminus U$ is a smooth manifold without boundary and $W \setminus U \subset M \setminus U$ is a smooth submanifold without boundary and if there is some subset $V \subset M$ such that*

$$\overline{U} \subset \mathring{V} \subset \overline{V} \subset W,$$

then the inclusion $(M \setminus U, W \setminus U) \hookrightarrow (M, W)$ induces an isomorphism

$$\mathcal{H}_*(M \setminus U, W \setminus U) \cong \mathcal{H}_*(M, W).$$

Definition (3.40). A topological space X is called **normal** if disjoint closed sets $A, B \subset X$ can be separated by open sets, i.e., there are disjoint open sets $U, V \subset X$ with $A \subset U$ and $B \subset V$. $\diamond$

Remark (3.41). All metric spaces are normal. In particular, every smooth manifold is normal (since every smooth manifold can be given a Riemannian structure, and thus yielding a metric). $\diamond$

Corollary (3.42). *Let M be a smooth manifold without boundary and $W \subset M$ an open subset. If $U \subset M$ is a closed subset with $U \subset W$, then the inclusion $(M \setminus U, W \setminus U) \hookrightarrow (M, W)$ induces an isomorphism*

$$\mathcal{H}_* (M \setminus U, W \setminus U) \cong \mathcal{H}_* (M, W).$$

Proof. Since $W \subset M$ is open, W is a smooth submanifold without boundary. Moreover, $M \setminus U$ is a smooth manifold without boundary and the subset $W \setminus U \subset M \setminus U$ is a smooth submanifold without boundary because U is closed.

Thus it remains to show that there is an open subset $V \subset M$ with

$$U \subset V \subset \overline{V} \subset W.$$

Since M is normal and $U \subset W$, we can separate the disjoint closed sets U and $M \setminus W$ by open sets V and V'. In particular,

$$U \subset V \subset \overline{V} \subset M \setminus V' \subset W.$$

Hence excision proves the corollary. $\square$

Corollary (3.43) (Mayer-Vietoris Sequence). *Suppose M is a smooth manifold without boundary and $U, V \subset M$ are open subsets with*

$$U \cup V = M.$$

Then the sequence

$$\cdots \longrightarrow \mathcal{H}_{k+1} (U \cap V) \xrightarrow{A} \mathcal{H}_{k+1} (U) \oplus \mathcal{H}_{k+1} (V) \xrightarrow{B} \mathcal{H}_{k+1} (M) \xrightarrow{\Delta} \mathcal{H}_k (U \cap V) \longrightarrow \cdots$$

is exact for all $k \in \mathbf{N}$, where

$$\begin{aligned} j_U &: U \cap V \hookrightarrow U, & i_U &: U \hookrightarrow M, \\ j_V &: U \cap V \hookrightarrow V, & i_V &: V \hookrightarrow M \end{aligned}$$

are the inclusion maps,

$$A := \big(\mathcal{H}_{k+1}(j_U), -\mathcal{H}_{k+1}(j_V)\big),$$
$$B := \big(\mathcal{H}_{k+1}(i_U), \mathcal{H}_{k+1}(i_V)\big),$$

and Δ *is defined below.*

Proof. By Corollary (3.42) the inclusion $(V, U \cap V) \hookrightarrow (M, U)$ induces an isomorphism $\mathcal{H}_*(V, U \cap V) \cong \mathcal{H}_*(M, U)$. We define Δ by the composition

$$\mathcal{H}_{k+1}(M) \longrightarrow \mathcal{H}_{k+1}(M, U) \xleftarrow{\cong} \mathcal{H}_{k+1}(V, U \cap V) \longrightarrow \mathcal{H}_k(U \cap V)$$

where the arrow on the left is induced by the inclusion $M \hookrightarrow (M, U)$ and the arrow on the right is the connecting homomorphism of the long exact measure homology sequence of the pair $(V, U \cap V)$.

The long exact measure homology sequences of the pairs $(V, U \cap V)$ and (M, U) yield a commutative ladder

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{H}_{k+1}(U \cap V) \arrow[r] \arrow[d] & \mathcal{H}_{k+1}(V) \arrow[r] \arrow[d] & \mathcal{H}_{k+1}(V, U \cap V) \arrow[r] \arrow[d, "\cong"'] & \mathcal{H}_k(U \cap V) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathcal{H}_{k+1}(U) \arrow[r] & \mathcal{H}_{k+1}(M) \arrow[r] \arrow[urr, bend right, "\Delta"'] & \mathcal{H}_{k+1}(M, U) \arrow[r] & \mathcal{H}_k(U) \arrow[r] & \cdots
\end{tikzcd}$$

with exact rows. Thus the sequence in the theorem is exact [tom Dieck, Lemma (8.3) in Kapitel IV]. ☐

Moreover, measure homology is compatible with colimits.

Theorem (3.44). *Let M be a smooth manifold without boundary covered by an ascending family $(U_k)_{k \in \mathbf{N}}$ of open subsets. Then the inclusions $U_k \hookrightarrow M$ induce an isomorphism*

$$\operatorname*{colim}_{k \to \infty} \mathcal{H}_*(U_k) \cong \mathcal{H}_*(M).$$

The idea of the proof is essentially the same as in singular homology: to use the compactness of the standard simplex. However, we have to be careful about some technicalities. The key observation is the following:

Lemma (3.45). *Let M and $(U_k)_{k \in \mathbf{N}}$ be as in the theorem. Suppose $j \in \mathbf{N}$ and $\mu \in \mathcal{C}_j(M)$. Then we can find an index $k \in \mathbf{N}$ and a measure chain $\nu \in \mathcal{C}_j(U_k)$ such that*

$$\mu = \nu^{j_k} \qquad \text{and} \qquad \partial\mu = (\partial\nu)^{j_k},$$

where $j_k \colon \operatorname{map}_\infty(\Delta^j, U_k) \longrightarrow \operatorname{map}_\infty(\Delta^j, M)$ *is the map induced by the inclusion* $U_k \hookrightarrow M$. *Furthermore, if* μ *is a cycle, then so is* ν.

Proof (of Lemma (3.45)). By Corollary (3.26) the family $(\operatorname{im} j_k)_{k\in\mathbf{N}}$ is an open ascending covering of $\operatorname{map}_\infty(\Delta^j, M)$, using the notation

$$\forall_{k\in\mathbf{N}} \quad \operatorname{im} j_k := j_k\big(\operatorname{map}_\infty(\Delta^j, U_k)\big).$$

Let D be a compact determination set for μ. Hence there is an index $k \in \mathbf{N}$ such that $D \subset j_k\big(\operatorname{map}_\infty(\Delta^j, U_k)\big)$.

Let $\mathcal{A}_k$ be the Borel σ-algebra of $\operatorname{im} j_k$ (with the subspace topology). By Lemma (3.25) the set $\operatorname{im} j_k$ is open in $\operatorname{map}_\infty(\Delta^j, M)$. Thus $\mathcal{A}_k$ is contained in the Borel σ-algebra of $\operatorname{map}_\infty(\Delta^j, M)$. Therefore we may define the signed measure $\widetilde{\mu}$ on $\operatorname{im} j_k$ via

$$\forall_{A\in\mathcal{A}_k} \quad \widetilde{\mu}(A) := \mu(A).$$

By construction, $D \subset \operatorname{im} j_k$, implying that D is a compact determination set of $\widetilde{\mu}$. Using Lemma (3.4), it is easy to see that $\widetilde{\mu}$ has finite total variation.

If $J_k \colon \operatorname{im} j_k \longrightarrow \operatorname{map}_\infty(\Delta^j, U_k)$ is the inverse of j_k, then J_k is continuous by Lemma (3.25). Hence the push-forward

$$\nu := \widetilde{\mu}^{J_k}$$

is a signed measure on $\operatorname{map}_\infty(\Delta^j, U_k)$ having a compact determination set and finite total variation (Lemma (3.9)), i.e., $\nu \in \mathcal{C}_j\,(U_k)$.

Our next step is to show the equality $\mu = \nu^{j_k}$: Let $A \subset \operatorname{map}_\infty(\Delta^j, M)$ be some Borel set. Hence

$$\widetilde{A} := A \cap \operatorname{im} j_k \in \mathcal{A}_k$$

and

$$\mu(A) = \mu(\widetilde{A}) + \mu\big(A \cap (\operatorname{map}_\infty(\Delta^j, M) \setminus \operatorname{im} j_k)\big).$$

Since $\operatorname{map}_\infty(\Delta^j, M) \setminus \operatorname{im} j_k$ is a measurable set contained in the complement of the determination set D,

$$\mu(A) = \mu(\widetilde{A})$$

holds. Therefore the constructions of j_k and J_k yield

$$\begin{aligned}
\mu(A) &= \mu(\widetilde{A}) &&= \widetilde{\mu}(\widetilde{A}) \\
&= \widetilde{\mu}\big(J_k^{-1}\big(j_k^{-1}(\widetilde{A})\big)\big) &&= \widetilde{\mu}^{J_k}\big(j_k^{-1}(A \cap \operatorname{im} j_k)\big) \\
&= \nu\big(j_k^{-1}(A \cap \operatorname{im} j_k)\big) &&= \nu\big(j_k^{-1}(A)\big) \\
&= \nu^{j_k}(A).
\end{aligned}$$

Since j_k is induced by the (smooth) inclusion $U_k \hookrightarrow M$, the map $\cdot^{j_k}$ is a chain map. In particular,

$$\partial\mu = \partial(\nu^{j_k}) = (\partial\nu)^{j_k}.$$

Suppose μ is a cycle and $A \subset \operatorname{map}_\infty(\Delta^j, U_k)$ is a Borel set. Then $j_k(A) = J_k^{-1}(A)$ lies in $\mathcal{A}_k$. As mentioned above, this results in $j_k(A)$ being Borel in $\operatorname{map}_\infty(\Delta^j, M)$. Using the injectivity of j_k we therefore compute

$$\partial\nu(A) = \partial\nu\big(j_k^{-1}(j_k(A))\big) = (\partial\nu)^{j_k}\big(j_k(A)\big) = \partial\mu\big(j_k(A)\big) = 0.$$

Hence ν is also a cycle. □

Proof (of Theorem (3.44)). For each $k \in \mathbf{N}$ and each $\ell \in \mathbf{N}_{\geq k}$ we write

$$i_k \colon U_k \hookrightarrow M \quad \text{and} \quad i_k^\ell \colon U_k \hookrightarrow U_\ell$$

for the inclusions. Let $j \in \mathbf{N}$. Then $\big(\mathcal{H}_j\,(U_k)\big)_{k\in\mathbf{N}}$ becomes a direct system with respect to the structure maps $\big(\mathcal{H}_j(i_k^\ell)\big)_{(k,\ell)\in D}$ where

$$D := \{(k,\ell) \in \mathbf{N}^2 | \ell \geq k\}.$$

Hence we can form the direct limit $\operatorname{colim}_{k\to\infty} \mathcal{H}_j\,(U_k)$ of this direct system.

The universal property of the direct limit provides us with a homomorphism

$$h_j \colon \operatorname*{colim}_{k\to\infty} \mathcal{H}_j\,(U_k) \longrightarrow \mathcal{H}_j\,(M)$$

induced by the maps $\big(\mathcal{H}_j\,(i_k)\big)_{k\in\mathbf{N}}$. To see why h_j happens to be an isomorphism we use the following fact [Bredon, Corollary D.3]: it is sufficient to show that for each $\alpha \in \mathcal{H}_j\,(M)$ there exists a $k \in \mathbf{N}$ and a $\beta \in \mathcal{H}_j\,(U_k)$ such that $\mathcal{H}_j\,(i_k)\beta = \alpha$, and that for each $\alpha \in \mathcal{H}_j\,(U_k)$ with $\mathcal{H}_j\,(i_k)(\alpha) = 0$ there is an $\ell \in \mathbf{N}$ such that $\mathcal{H}_j\,\big(i_k^\ell\big)(\alpha) = 0$.

1. Let $\mu \in \mathcal{C}_j\,(M)$ be a measure cycle. Then the preceding lemma provides us with some $k \in \mathbf{N}$ and a measure cycle $\nu \in \mathcal{C}_j\,(U_k)$ such that $\mu = \nu^{j_k}$. This proves
$$\mathcal{H}_j\,(i_k)\big([\nu]\big) = [\mu]$$
in $\mathcal{H}_j\,(M)$, since j_k is induced by the inclusion i_k.

2. Let $k \in \mathbf{N}$ and $\nu \in \mathcal{C}_j\,(U_k)$ be a measure cycle such that $\mathcal{H}_j\,(i_k)\big([\nu]\big) = 0 \in \mathcal{H}_j\,(M)$. Hence there exists a measure chain $\tau \in \mathcal{C}_{j+1}\,(M)$ with
$$\partial\tau = \nu^{j_k}.$$

The above lemma gives us an $\ell \in \mathbf{N}$ (and the proof shows that we can assume $\ell \geq k$) and a measure chain $\varrho \in \mathcal{C}_{j+1}(U_\ell)$ satisfying

$$\begin{aligned} \varrho^{j_\ell} &= \tau, \\ (\partial\varrho)^{j_\ell} &= \partial\tau = \nu^{j_k}. \end{aligned}$$

Writing $j_k^\ell \colon \operatorname{map}_\infty(\Delta^j, U_k) \longrightarrow \operatorname{map}_\infty(\Delta^j, U_\ell)$ for the map induced by the inclusion i_k^ℓ, we conclude

$$(\partial\varrho)^{j_\ell} = \nu^{j_k} = \big(\nu^{j_k^\ell}\big)^{j_\ell}.$$

As in the last part of the proof of Lemma (3.45), we can compute for all Borel sets $A \subset \operatorname{map}_\infty(\Delta^j, U_k)$

$$\begin{aligned} \partial\varrho(A) &= \partial\varrho\big(j_\ell^{-1}(j_\ell(A))\big) = (\partial\varrho)^{j_\ell}\big(j_\ell(A)\big) \\ &= \big(\nu^{j_k^\ell}\big)^{j_\ell}\big(j_\ell(A)\big) \quad = \nu^{j_k^\ell}\big(j_\ell^{-1}(j_\ell(A))\big) \\ &= \nu^{j_k^\ell}(A). \end{aligned}$$

This shows $\mathcal{H}_j\left(i_k^\ell\right)\big([\nu]\big) = 0$ in $\mathcal{H}_j\left(U_\ell\right)$. $\square$

With the help of these algebraic properties we will see in Section 4.2 that measure homology and singular homology (algebraically) coincide.

4 Measure Homology is Singular Homology

In this chapter, singular homology (with real coefficients) and measure homology of smooth manifolds without boundary are compared. The results from Section 3.3 already suggest that the homology groups of these two theories should coincide. Surprisingly, the natural isomorphism providing this correspondence turns out to be *isometric*. It is thus possible to calculate the simplicial volume of an oriented smooth closed connected manifold via measure homology.

In order to obtain the mentioned isometric isomorphism between singular homology and measure homology, we proceed in three steps:

- First of all, we introduce smooth singular homology (for smooth manifolds without boundary) and show that it is canonically isometrically isomorphic to singular homology.
- In the second step, we establish an (algebraic) isomorphism between smooth singular homology and measure homology. This is done by means of the properties described in Section 3.3.
- Thirdly, we prove that this isomorphism is actually isometric. This is the central contribution of this thesis. The method of proof is inspired by the duality principle of bounded cohomology (Theorem (2.7)).

In the last section, we discuss an integration operation for measure chains as a tool to analyse the top homology group, which will turn out to be useful in the context of smearing (Section 5.4).

4.1 ■ Smooth singular homology

Since measure homology is based on smooth simplices, it is easier to see a connection between measure homology and smooth singular homology instead of singular homology. Smooth singular homology is defined just like ordinary singular homology, using smooth singular chains instead of continuous ones.

Definition (4.1). Let M be a smooth manifold without boundary and $k \in \mathbf{N}$. The elements of $\operatorname{map}_\infty(\Delta^k, M)$ are called **smooth singular k-simplices of** M. The subcomplex (using the inclusion from Remark (3.17))

$$\big(C_*^{\mathrm{sm}}(M,\mathbf{R}), \partial|_{C_*^{\mathrm{sm}}(M,\mathbf{R})}\big) \subset \big(C_*(M,\mathbf{R}), \partial\big)$$

generated by all smooth singular simplices yields the **smooth singular homology groups**

$$H_k^{\mathrm{sm}}(M,\mathbf{R}) := \frac{\ker\big(\partial\colon C_k^{\mathrm{sm}}(M,\mathbf{R}) \longrightarrow C_{k-1}^{\mathrm{sm}}(M,\mathbf{R})\big)}{\operatorname{im}\big(\partial\colon C_{k+1}^{\mathrm{sm}}(M,\mathbf{R}) \longrightarrow C_k^{\mathrm{sm}}(M,\mathbf{R})\big)}$$

for each $k \in \mathbf{N}$. ◇

More details about smooth singular homology can be found in the textbooks of Massey and Lee [Massey, §2 in Appendix A], [Lee$_2$, Chapter 16]. In Massey's book it is shown (using the same methods as in ordinary singular homology) that smooth singular homology satisfies theorems corresponding to those stated in Section 3.3. These statements will in the following be referred to via numbers such as $(3.39)^{\mathrm{sm}}$ instead of (3.39).

Moreover, by the same pattern as for singular homology, we can define a norm $\|\cdot\|_1^{\mathrm{sm}}$ on the smooth singular chain complex and its induced semi-norm $\|\cdot\|_1^{\mathrm{sm}}$ on smooth singular homology.

Theorem (4.2). *For every smooth manifold M without boundary the natural chain complex inclusion*

$$j_M\colon C_*^{\mathrm{sm}}(M,\mathbf{R}) \hookrightarrow C_*(M,\mathbf{R})$$

induces an isometric isomorphism between the homology groups $H_*^{\mathrm{sm}}(M,\mathbf{R})$ and $H_*(M,\mathbf{R})$.

Proof. Via the Whitney approximation theorem, a smoothing operator

$$s\colon C_*(M,\mathbf{R}) \longrightarrow C_*^{\mathrm{sm}}(M,\mathbf{R})$$

can be constructed [Lee$_2$, page 417], satisfying the following conditions:

- The map s is a chain map with
$$s \circ j_M = \mathrm{id} \quad \text{and} \quad j_M \circ s \simeq \mathrm{id}\,.$$
- For each singular simplex $\sigma \in S_k(M)$ the image $s(\sigma) \in C_k^{\mathrm{sm}}(M,\mathbf{R})$ consists of just one smooth simplex.

The first part implies that $H_*(j_M)\colon H_*^{\mathrm{sm}}(M,\mathbf{R}) \longrightarrow H_*(M,\mathbf{R})$ is an isomorphism. Its inverse is $H_*(s)$.

From the second part we deduce: if $\sum_{\sigma\in S_k(M)} a_\sigma \cdot \sigma \in C_k(M,\mathbf{R})$ is a cycle representing the homology class $\alpha \in H_k(M,\mathbf{R})$, then $\sum_{\sigma\in S_k(M)} a_\sigma \cdot s(\sigma)$ is a cycle representing $H_k(s)(\alpha)$ with

$$\begin{aligned} \|H_k(s)(\alpha)\|_1^{\mathrm{sm}} &\le \Big\| \sum_{\sigma\in S_k(M)} a_\sigma \cdot s(\sigma) \Big\|_1^{\mathrm{sm}} \le \sum_{\sigma\in S_k(M)} |a_\sigma| \\ &= \Big\| \sum_{\sigma\in S_k(M)} a_\sigma \cdot \sigma \Big\|_1 . \end{aligned}$$

Taking the infimum over all representatives of α thus yields

$$\|H_k(s)(\alpha)\|_1^{\mathrm{sm}} \le \|\alpha\|_1 = \big\|H_k(j_M)\big(H_k(s)(\alpha)\big)\big\|_1 .$$

On the other hand, obviously

$$\big\|H_k(j_M)(\alpha')\big\|_1 \le \|\alpha'\|_1^{\mathrm{sm}}$$

holds for all $\alpha' \in H_k^{\mathrm{sm}}(M,\mathbf{R})$. Since $H_k(s)$ is a bijection this proves the theorem. □

In particular, we detect the fundamental class in smooth singular homology and we can compute the simplicial volume via this class:

Definition (4.3). Let M be an oriented smooth closed connected manifold of dimension n. The above theorem implies that

$$[M]_{\mathrm{sm}} := \big(H_n(j_M)\big)^{-1}\big([M]_{\mathbf{R}}\big)$$

is a well-defined generator of $H_n^{\mathrm{sm}}(M,\mathbf{R})$. ◇

Corollary (4.4). *Let M be an oriented smooth closed connected manifold. Then*

$$\big\|[M]_{\mathrm{sm}}\big\|_1^{\mathrm{sm}} = \|M\|$$

holds by the above theorem. □

In Section 4.3, we will make use of a version of bounded cohomology based on smooth singular homology. Its construction is completely analogous to the corresponding definition of bounded (singular) cohomology and it turns out to be the same as ordinary bounded cohomology. However, bounded smooth cohomology is more feasible in the context of measure homology.

Definition (4.5). Let M be a smooth manifold without boundary, and $k \in \mathbf{N}$.

- The supremum norm of $f \in \mathrm{Hom}_{\mathbf{R}}\big(C_k^{\mathrm{sm}}(M, \mathbf{R}), \mathbf{R}\big)$ is given by

  $$\|f\|_\infty := \sup_{\sigma \in \mathrm{map}_\infty(\Delta^k, M)} |f(\sigma)|.$$

- We write

  $$\widehat{C}_{\mathrm{sm}}^k(M) := \big\{f \in \mathrm{Hom}_{\mathbf{R}}\big(C_k^{\mathrm{sm}}(M, \mathbf{R}), \mathbf{R}\big) \bigm| \|f\|_\infty < \infty\big\}$$

  for the group of **bounded smooth k-cochains**.

- The corresponding coboundary operator is defined via

  $$\begin{aligned} \delta\colon \widehat{C}_{\mathrm{sm}}^k(M) &\longrightarrow \widehat{C}_{\mathrm{sm}}^{k+1}(M) \\ f &\longmapsto \big(c \mapsto (-1)^{k+1} \cdot f(\partial(c))\big). \end{aligned}$$

- The **k-th bounded smooth cohomology group $\widehat{H}_{\mathrm{sm}}^k(M)$ of M** is the k-th cohomology group of the cochain complex $(\widehat{C}_{\mathrm{sm}}^*(M), \delta)$.

- The supremum norm induces (as in the case of ordinary bounded cohomology) a seminorm on the quotient $\widehat{H}_{\mathrm{sm}}^k(M)$, which will also be denoted by $\|\cdot\|_\infty$. ◇

Remark (4.6). As in the case of ordinary bounded cohomology, it is easy to verify that the homomorphism δ is well-defined and satisfies $\delta \circ \delta = 0$. □

Using the smoothing operator s of the proof of Theorem (4.2), it can readily be seen that $\widehat{H}_{\mathrm{sm}}^*(M)$ and $\widehat{H}^*(M)$ are isometrically isomorphic:

Theorem (4.7). *Let M be a smooth manifold without boundary. Then the restriction map (where j_M is defined as in Theorem (4.2))*

$$\begin{aligned} \widehat{C}^*(j_M) \colon \widehat{C}^*(M) &\longrightarrow \widehat{C}^*_{\mathrm{sm}}(M) \\ f &\longmapsto f|_{C_*^{\mathrm{sm}}(M,\mathbf{R})} = f \circ j_M \end{aligned}$$

induces an isometric isomorphism $\widehat{H}^*(j_M)\colon \widehat{H}^*(M) \longrightarrow \widehat{H}^*_{\mathrm{sm}}(M)$ on cohomology.

Proof. It is clear that the restriction map is a well-defined cochain map.

Let $h_* \colon C_*(M,\mathbf{R}) \longrightarrow C_{*+1}(M,\mathbf{R})$ be a chain homotopy $j_M \circ s \simeq \mathrm{id}$. The same arguments as in the proof of Lemma (2.5) yield that the "dual"

$$\begin{aligned} \widehat{C}^*(M) &\longrightarrow \widehat{C}^{*-1}(M) \\ f &\longmapsto (-1)^* \cdot f \circ h_{*-1} \end{aligned}$$

is well-defined and a cochain homotopy $\widehat{C}^*(s) \circ \widehat{C}^*(j_M) \simeq \mathrm{id}$, where $\widehat{C}^*(s)$ and $\widehat{C}^*(j_M)$ are the "duals" of s and j_M respectively. Moreover,

$$\widehat{C}^*(j_M) \circ \widehat{C}^*(s) = \mathrm{id},$$

by functoriality. Hence the restriction map induces an isomorphism on cohomology.

Since s maps singular simplices to singular simplices, we obtain

$$\|f \circ s\|_\infty \leq \|f\|_\infty$$

for all $f \in \widehat{C}^*_{\mathrm{sm}}(M)$. On the other hand, obviously $\|f \circ j_M\|_\infty \leq \|f\|_\infty$ holds for all $f \in \widehat{C}^*(M)$. Therefore we conclude

$$\forall_{\varphi \in \widehat{H}^*(M)} \quad \|\varphi\|_\infty = \big\|\widehat{H}^*(s)\big(\widehat{H}^*(j_M)(\varphi)\big)\big\|_\infty \leq \big\|\widehat{H}^*(j_M)(\varphi)\big\|_\infty \leq \|\varphi\|_\infty,$$

which implies that $\widehat{H}^*(j_M)$ is isometric. ☐

As in the case of ordinary bounded cohomology, we can define a **Kronecker product**

$$\langle\,\cdot\,,\cdot\,\rangle \colon \widehat{C}^*_{\mathrm{sm}}(M) \times C^{\mathrm{sm}}_*(M,\mathbf{R}) \longrightarrow \mathbf{R}.$$

by evaluation. Analogously, this product descends to a bilinear map

$$\langle\,\cdot\,,\cdot\,\rangle \colon \widehat{H}^*_{\mathrm{sm}}(M) \times H^{\mathrm{sm}}_*(M,\mathbf{R}) \longrightarrow \mathbf{R},$$

and it is readily seen to be compatible with the ordinary Kronecker product, i.e., for all $\varphi \in \widehat{H}^*(M)$ and all $\alpha \in H_*^{\mathrm{sm}}(M, \mathbf{R})$

$$\left\langle \widehat{H}^*(j_M)(\varphi), \alpha \right\rangle = \left\langle \varphi, H_*(j_M)(\alpha) \right\rangle.$$

We now obtain a smooth version of the duality principle (compare Theorem (2.7)), which will become useful in Section 4.3.

Corollary (4.8) (Duality Principle of Bounded Smooth Cohomology). *Let M be a smooth manifold without boundary and let $k \in \mathbf{N}$ and $\alpha \in H_k^{\mathrm{sm}}(M, \mathbf{R})$.*

1. *Then $\|\alpha\|_1^{\mathrm{sm}} = 0$ if and only if*

$$\forall_{\varphi \in \widehat{H}_{\mathrm{sm}}^k(M)} \quad \langle \varphi, \alpha \rangle = 0.$$

2. *If $\|\alpha\|_1^{\mathrm{sm}} > 0$, then*

$$\|\alpha\|_1^{\mathrm{sm}} = \sup\left\{ \frac{1}{\|\varphi\|_\infty} \;\middle|\; \varphi \in \widehat{H}_{\mathrm{sm}}^k(M), \langle \varphi, \alpha \rangle = 1 \right\}.$$

Proof. By the above remark,

$$\begin{aligned} \langle \varphi, \alpha \rangle &= \left\langle \widehat{H}^k(j_M) \circ \widehat{H}^k(j_M)^{-1}(\varphi), \alpha \right\rangle \\ &= \left\langle \widehat{H}^k(j_M)^{-1}(\varphi), H_k(j_M)(\alpha) \right\rangle \end{aligned}$$

holds for all $\alpha \in H_k^{\mathrm{sm}}(M, \mathbf{R})$ and all $\varphi \in \widehat{H}^k(M)$. Since $\widehat{H}^k(j_M)$ and $H_k(j_M)$ are isometric isomorphisms by Theorem (4.7) and Theorem (4.2), the duality principle (2.7) applied to $H_k(j_M)(\alpha) \in H_k(M, \mathbf{R})$ proves the corollary. ☐

4.2 ■ The algebraic isomorphism

In this section, smooth singular homology (with real coefficients) and measure homology are shown to be isomorphic as vector spaces. This isomorphism is induced by a natural chain map:

Lemma (4.9). *Let M be a smooth manifold without boundary. Then there is a norm-preserving injective chain map*

$$i_M \colon C_*^{\mathrm{sm}}(M, \mathbf{R}) \longrightarrow \mathcal{C}_*(M).$$

Proof. Let $k \in \mathbf{N}$. For $\sigma \in \operatorname{map}_\infty(\Delta^k, M)$ we denote the atomic measure on $\operatorname{map}_\infty(\Delta^k, M)$ concentrated in σ by δ_σ, which is obviously an element of $\mathcal{C}_k(M)$. If $c := \sum_{\sigma\in\operatorname{map}_\infty(\Delta^k,M)} a_\sigma \cdot \sigma$ is a smooth chain, we define

$$i_M(c) := \sum_{\sigma\in\operatorname{map}_\infty(\Delta^k,M)} a_\sigma \cdot \delta_\sigma \in \mathcal{C}_k(M).$$

The calculation showing that i_M is a chain map is quite straightforward [Ratcliffe, Lemma 2 in §11.5].

It remains to show that i_M is norm-preserving. With the above notation we obtain that the Hahn decomposition of $i_M(c)$ is given by

$$\begin{aligned} P &:= \big\{\sigma \in \operatorname{map}_\infty(\Delta^k, M) \bigm| a_\sigma \geq 0\big\}, \\ N &:= \big\{\sigma \in \operatorname{map}_\infty(\Delta^k, M) \bigm| a_\sigma < 0\big\}. \end{aligned}$$

Hence we conclude

$$\begin{aligned} \|i_M(c)\| &= |i_M(c)|\big(\operatorname{map}_\infty(\Delta^k, M)\big) \\ &= i_M(c)(P) - i_M(c)(N) \\ &= \sum_{\sigma\in P} a_\sigma - \sum_{\sigma\in N} a_\sigma \\ &= \sum_{\sigma\in\operatorname{map}_\infty(\Delta^k,M)} |a_\sigma| \\ &= \|c\|_1^{\mathrm{sm}}. \end{aligned}$$ □

Theorem (4.10). *If M is a smooth manifold without boundary, then the inclusion $i_M\colon C_*^{\mathrm{sm}}(M, \mathbf{R}) \hookrightarrow \mathcal{C}_*(M)$ of the previous lemma induces an isomorphism*

$$H_*^{\mathrm{sm}}(M, \mathbf{R}) \cong \mathcal{H}_*(M).$$

In the case of measure homology based on the compact open topology (instead of the $\mathcal{C}^1$-topology), proofs of this theorem were given by Hansen and Zastrow via the verification of the Eilenberg-Steenrod axioms [Hansen, Theorem 1.1], [Zastrow, Theorem 3.4]. In the smooth setting however, it is not entirely clear how to use this approach since only smooth manifolds can be handled (instead of the whole universe of CW-complexes).

Therefore we give a proof based on some "manifold induction" – following a general idea of Milnor:

Theorem (4.11). *Assume P is some property of smooth manifolds without boundary satisfying the following conditions:*

1. *The property P is stable under smooth homotopy equivalences.*

2. *The property P holds for the one point space (which can be thought of as a zero-dimensional smooth manifold without boundary).*

3. *If U, V are open subsets of the same smooth manifold without boundary such that U, V and $U \cap V$ have property P, then the union $U \cup V$ also has property P.*

4. *If $(U_k)_{k\in\mathbf{N}}$ is an ascending sequence of subsets of some (common) smooth manifold without boundary all having property P, then $\bigcup_{k\in\mathbf{N}} U_k$ satisfies P.*

Then each smooth manifold without boundary has property P.

Proof. The (not very difficult) proof can for example be found in Massey's book [Massey, Case 5 on page 364]. ☐

Proof (of Theorem (4.10)). We consider the property P which is defined as follows: a smooth manifold U without boundary satisfies P if and only if the homomorphism

$$H_*(i_U)\colon H_*^{\mathrm{sm}}(U,\mathbf{R}) \longrightarrow \mathcal{H}_*(U)$$

is an isomorphism.

In view of Theorem (4.11) it suffices to show that P has the four mentioned properties:

1. Suppose $f\colon M \longrightarrow N$ is a smooth homotopy equivalence of smooth manifolds without boundary where M enjoys P. Then the diagram

$$\begin{tikzcd} H_*^{\mathrm{sm}}(M,\mathbf{R}) \arrow[r, "H_*^{\mathrm{sm}}(f{,}\mathbf{R})"] \arrow[d, "H_*(i_M)"'] & H_*^{\mathrm{sm}}(N,\mathbf{R}) \arrow[d, "H_*(i_N)"] \\ \mathcal{H}_*(M) \arrow[r, "\mathcal{H}_*(f)"'] & \mathcal{H}_*(N) \end{tikzcd}$$

is commutative and all arrows but the right vertical one are isomorphisms. Hence $H_*(i_N)$ must also be an isomorphism, showing that N fulfils P.

2. The chain complexes $\mathcal{C}_*(\bullet)$ and $C_*^{\mathrm{sm}}(\bullet,\mathbf{R})$ are identical and the inclusion $i_\bullet\colon \mathcal{C}_*(\bullet) \hookrightarrow C_*^{\mathrm{sm}}(\bullet,\mathbf{R})$ is the identity. Hence the one point space satisfies P.

3. Let U, V be open subsets of some smooth manifold M without boundary such that U, V, and $U \cap V$ satisfy P. Then we obtain the following commutative ladder

$$\begin{tikzcd}[column sep=small]
H^{\mathrm{sm}}_{k+1}(U \cap V, \mathbf{R}) \arrow[r] \arrow[d, "H_{k+1}(i_{U\cap V})"', "\cong"] & \begin{matrix} H^{\mathrm{sm}}_{k+1}(U,\mathbf{R}) \\ \oplus \\ H^{\mathrm{sm}}_{k+1}(V,\mathbf{R}) \end{matrix} \arrow[r] \arrow[d, "\begin{matrix} H_{k+1}(i_u) \\ \oplus H_{k+1}(i_V) \end{matrix}"', "\cong"] & H^{\mathrm{sm}}_{k+1}(U \cup V, \mathbf{R}) \arrow[r] \arrow[d, "H_{k+1}(i_{U\cup V})"'] & H^{\mathrm{sm}}_{k}(U \cap V, \mathbf{R}) \arrow[r] \arrow[d, "H_k(i_{U\cap V})"', "\cong"] & \begin{matrix} H^{\mathrm{sm}}_{k}(U,\mathbf{R}) \\ \oplus \\ H^{\mathrm{sm}}_{k}(V,\mathbf{R}) \end{matrix} \arrow[d, "H_k(i_U)\oplus H_k(i_V)"', "\cong"] \\
\mathcal{H}_{k+1}(U \cap V) \arrow[r] & \begin{matrix} \mathcal{H}_{k+1}(U) \\ \oplus \\ \mathcal{H}_{k+1}(V) \end{matrix} \arrow[r] & \mathcal{H}_{k+1}(U \cup V) \arrow[r] & \mathcal{H}_{k}(U \cap V) \arrow[r] & \begin{matrix} \mathcal{H}_{k}(U) \\ \oplus \\ \mathcal{H}_{k}(V) \end{matrix}
\end{tikzcd}$$

whose rows are Mayer-Vietoris sequences and thus exact (compare Corollary (3.43) and $(3.43)^{\mathrm{sm}}$). Now the five lemma implies that the union $U \cup V$ has property P.

4. Let M be a smooth manifold without boundary and $(U_k)_{k\in\mathbf{N}}$ an ascending sequence of open subsets of M. Suppose that each U_k satisfies P.

Since $\bigcup_{k\in\mathbf{N}} U_K$ is a smooth manifold without boundary covered by the ascending family $(U_k)_{k\in\mathbf{N}}$ of open sets, we conclude from Theorem (3.44) and Theorem $(3.44)^{\mathrm{sm}}$ that the inclusions induce isomorphisms

$$i\colon \operatorname*{colim}_{k\to\infty} \mathcal{H}_*(U_k) \longrightarrow \mathcal{H}_*(U),$$
$$j\colon \operatorname*{colim}_{k\to\infty} H^{\mathrm{sm}}_*(U_k, \mathbf{R}) \longrightarrow H^{\mathrm{sm}}_*(U, \mathbf{R})$$

using the notation

$$U := \bigcup_{k\in\mathbf{N}} U_k.$$

Furthermore, for each $k \in \mathbf{N}$ the diagram

$$\begin{tikzcd}
H^{\mathrm{sm}}_*(U_k, \mathbf{R}) \arrow[r, "H^{\mathrm{sm}}_*(U_k \hookrightarrow U, \mathbf{R})"] \arrow[d, "H_*(i_{U_k})"'] & H^{\mathrm{sm}}_*(U, \mathbf{R}) \arrow[d, "H_*(i_U)"] \\
\mathcal{H}_*(U_k) \arrow[r, "\mathcal{H}_*(U_k \hookrightarrow U)"'] & \mathcal{H}_*(U)
\end{tikzcd}$$

commutes. Since each U_k has property P, we conclude that the induced homomorphism

$$\operatorname*{colim}_{k\to\infty} H_*(i_{U_k})\colon \operatorname*{colim}_{k\to\infty} H^{\mathrm{sm}}_*(U_k, \mathbf{R}) \longrightarrow \operatorname*{colim}_{k\to\infty} \mathcal{H}_*(U_k)$$

is an isomorphism. The previous diagram shows that

$$\begin{tikzcd}
\operatorname{colim}_{k\to\infty} H_*^{\mathrm{sm}}(U_k, \mathbf{R}) \arrow[r, "j"] \arrow[d, "\operatorname{colim}_{k\to\infty} H_*(i_{U_k})"'] & H_*^{\mathrm{sm}}(U, \mathbf{R}) \arrow[d, "H_*(i_U)"] \\
\operatorname{colim}_{k\to\infty} \mathcal{H}_*(U_k) \arrow[r, "i"'] & \mathcal{H}_*(U)
\end{tikzcd}$$

is commutative, implying that $H_*(i_U)$ is also an isomorphism. □

In particular, we obtain a fundamental class for measure homology. It is the aim of this chapter to show that this fundamental class can be used to compute the simplicial volume.

Definition (4.12). Let M be a smooth oriented closed connected manifold of dimension n. By Theorem (4.10) the **measure homology fundamental class**

$$[M]_{\mathrm{mh}} := H_n(i_M)\big([M]_{\mathrm{sm}}\big) \in \mathcal{H}_n(M)$$

is a generator of

$$\mathcal{H}_n(M) \cong H_n^{\mathrm{sm}}(M, \mathbf{R}) \cong H_n(M, \mathbf{R}) \cong \mathbf{R}. \qquad \diamond$$

4.3 ■ . . . is isometric

In this section it will be shown that the (algebraic) isomorphism from Theorem (4.10) is actually isometric. Thurston already stated that this should be true [Thurston, page 6.6], however the paper cited there has never been published in the suggested form. In early 2004, a preprint by Bowen appeared to remedy this situation [Bowen], but the given argument does not seem to be correct (see Subsection 4.3.4).

We recall the following notation which will be used throughout this section:

Definition (4.13). If M is a smooth manifold without boundary, we write $j_M\colon C_*^{\mathrm{sm}}(M, \mathbf{R}) \hookrightarrow C_*(M, \mathbf{R})$ for the obvious inclusion of chain complexes and $i_M\colon C_*^{\mathrm{sm}}(M, \mathbf{R}) \longrightarrow \mathcal{C}_*(M)$ for the injective norm-preserving chain map which is given by mapping smooth singular simplices to the corresponding atomic measure (compare (4.9)). ◇

Then the result can be formulated as follows:

Theorem (4.14). *If M is a smooth manifold without boundary, then the inclusion $i_M\colon C^{\mathrm{sm}}_*(M,\mathbf{R}) \hookrightarrow \mathcal{C}_*(M)$ induces an isometric isomorphism*

$$H^{\mathrm{sm}}_*(M,\mathbf{R}) \cong \mathcal{H}_*(M).$$

Before giving the proof we state the main consequence of this theorem:

Corollary (4.15). *Let M be a smooth manifold without boundary.*

1. *Then the inclusions i_M and j_M induce an isometric isomorphism*

$$H_*(M,\mathbf{R}) \cong \mathcal{H}_*(M).$$

2. *In particular, the simplicial volume of M can be computed via measure homology, i.e.,*

$$\|M\| = \big\|[M]_{\mathrm{mh}}\big\|_{\mathrm{mh}}. \qquad \square$$

Remark (4.16). The second part was already known to be true for hyperbolic manifolds [Gromov$_2$, page 235], [Ratcliffe, Exercise 10 in §11.5] but the general case was still open. $\diamond$

Remark (4.17). It seems possible to use the same proof in the case of measure homology for nice enough metric spaces (based on the compact open topology on the set of singular simplices), hence giving an isometric isomorphism between singular homology and measure homology for general (metric) spaces. $\diamond$

The proof of the theorem is based on the idea of imitating the duality results of Theorem (2.7) (and Corollary (4.8)). This yields a reformulation of the problem in terms of bounded cohomology, which can be solved by means of the established framework sketched in Section 2.3.

The drawback of this approach is that the proof of Theorem (4.14) is not as geometric as one might wish.

4.3.1 ■ A dual for measure homology

As a first step we have to construct the "dual" $\widehat{\mathcal{H}}^*(M)$ which plays the rôle of the bounded (smooth) cohomology groups $\widehat{H}^*_{\mathrm{sm}}(M)$ in the (smooth) singular theory. If $c = \sum_{\sigma\in\operatorname{map}_\infty(\Delta^k,M)} a_\sigma \cdot \sigma \in C^{\mathrm{sm}}_k(M,\mathbf{R})$ is a smooth singular chain and $f \in \widehat{C}^k_{\text{sm}}(M)$ is a singular cochain, their Kronecker product is given by

$$\langle f, c\rangle = f(c) = \sum_{\sigma \in \operatorname{map}_\infty(\Delta^k, M)} a_\sigma \cdot f(\sigma) \in \mathbf{R}.$$

If we think of c as a linear combination of atomic measures, this looks like an integration of f over c. Hence our "dual" in measure homology will consist of (bounded) functions which can be integrated over measure chains:

Definition (4.18). Let M be a smooth manifold without boundary and let $k \in \mathbf{N}$. We write

$$\widehat{\mathcal{C}}^k(M) := \bigl\{ f \colon \operatorname{map}_\infty(\Delta^k, M) \longrightarrow \mathbf{R} \bigm| f \text{ is Borel measurable and bounded} \bigr\}$$

and

$$\begin{aligned} \delta \colon \widehat{\mathcal{C}}^k(M) &\longrightarrow \widehat{\mathcal{C}}^{k+1}(M) \\ f &\longmapsto \bigl(\sigma \mapsto (-1)^{k+1} \cdot f(\partial(\sigma))\bigr). \end{aligned}$$

Here $f(\partial(\sigma))$ is an abbreviation for $\sum_{j=0}^{k+1} (-1)^j \cdot f(\sigma \circ \partial_j)$. ◇

The next lemma shows that this map $\delta \colon \widehat{\mathcal{C}}^k(M) \longrightarrow \widehat{\mathcal{C}}^{k+1}(M)$ is indeed well-defined and turns $\widehat{\mathcal{C}}^*(M)$ into a cochain complex.

Lemma (4.19). *Let M be a smooth manifold without boundary, $k \in \mathbf{N}$, and let $f \in \widehat{\mathcal{C}}^k(M)$. Then $\delta(f) \in \widehat{\mathcal{C}}^{k+1}(M)$ and*

$$\delta \circ \delta(f) = 0.$$

Proof. According to Remark (3.24),

$$\begin{aligned} \operatorname{map}_\infty(\Delta^{k+1}, M) &\longrightarrow \operatorname{map}_\infty(\Delta^k, M) \\ \sigma &\longmapsto \sigma \circ \partial_j \end{aligned}$$

is continuous for each $j \in \{0, \dots, k+1\}$. Hence $\delta(f)$ is Borel measurable. As f is bounded, $\delta(f)$ is bounded as well.

Since $\partial(\partial(\sigma)) = 0$ holds for all $\sigma \in \operatorname{map}_\infty(\Delta^{k+2}, M)$, it follows that

$$\delta \circ \delta(f) = 0.$$ □

Definition (4.20). Let M be a smooth manifold without boundary and let $k \in \mathbf{N}$. The ***k*-th bounded measure cohomology group of** M is given by

$$\widehat{\mathcal{H}}^k(M) := H^k\bigl(\widehat{\mathcal{C}}^*(M), \delta\bigr).$$

We write $\|\cdot\|_\infty$ for the seminorm on $\widehat{\mathcal{H}}^k(M)$ which is induced by the supremum norm on $\widehat{\mathcal{C}}^k(M)$. ◇

Definition (4.21). Let M be a smooth manifold without boundary and let $k \in \mathbf{N}$. The **Kronecker product** of $\mu \in \mathcal{C}_k(M)$ and $f \in \widehat{\mathcal{C}}^k(M)$ is defined as

$$\langle f, \mu \rangle := \int f \, d\mu.$$

If μ is a measure cycle and f is a cocycle, we write

$$\big\langle [f], [\mu] \big\rangle := \langle f, \mu \rangle = \int f \, d\mu. \qquad \diamond$$

It is easy to see that this is well-defined and – as desired – yields a generalisation of the ordinary Kronecker product:

Lemma (4.22). *Let M be a smooth manifold without boundary and let $k \in \mathbf{N}$.*

1. *The Kronecker product $\langle \cdot, \cdot \rangle \colon \widehat{\mathcal{C}}^k(M) \times \mathcal{C}_k(M) \longrightarrow \mathbf{R}$ is well-defined and bilinear.*
2. *The Kronecker product $\langle \cdot, \cdot \rangle \colon \widehat{\mathcal{H}}^k(M) \times \mathcal{H}_k(M) \longrightarrow \mathbf{R}$ on (co)homology is well-defined and bilinear.*
3. *The Kronecker product defined above is compatible with the Kronecker product on bounded smooth cohomology in the following sense: for all $f \in \widehat{\mathcal{C}}^k(M)$ and all $c \in C_k^{\mathrm{sm}}(M, \mathbf{R})$,*
$$\langle B(f), c \rangle = \langle f, i_M(c) \rangle,$$
where $B(f)$ denotes the linear extension of $f \colon \mathrm{map}_\infty(\Delta^k, M) \longrightarrow \mathbf{R}$ to the vector space $C_k^{\mathrm{sm}}(M, \mathbf{R})$. Passage to (co)homology yields for all $\varphi \in \widehat{\mathcal{H}}^k(M)$ and all $\alpha \in H_k^{\mathrm{sm}}(M, \mathbf{R})$
$$\big\langle H^k(B)(\varphi), \alpha \big\rangle = \big\langle \varphi, H_k(i_M)(\alpha) \big\rangle.$$

Proof. 1. The integral is defined and finite, since the elements of $\mathcal{C}_k(M)$ are (signed) measures of finite total variation and the elements of $\widehat{\mathcal{C}}^k(M)$ are bounded measurable functions. Moreover, the integral is obviously bilinear.

2. It remains to show that the definition of the Kronecker product on (co)homology does not depend on the chosen representatives. This is a consequence of the following fact: if $\ell \in \mathbf{N}$, $\mu \in \mathcal{C}_\ell(M)$ and $f \in \widehat{\mathcal{C}}^{\ell-1}(M)$, then the transformation formula yields

$$\int_{\mathrm{map}_\infty(\Delta^\ell, M)} f \circ \partial_j \, d\mu = \int_{\mathrm{map}_\infty(\Delta^{\ell-1}, M)} f \, d\mu^{(\sigma \mapsto \sigma \circ \partial_j)}$$

for all $j \in \{0, \dots, \ell\}$, and hence $\int \delta(f)\, d\mu = \int f\, d\partial(\mu)$.

3. Since both Kronecker products are bilinear, it suffices to consider the case where c consists of a single smooth simplex $\sigma \in \operatorname{map}_\infty(\Delta^k, M)$. Then the left hand side – by definition – evaluates to $f(\sigma)$. Since $i_M(c) = i_M(\sigma) = \delta_\sigma$ is the atomic measure on $\operatorname{map}_\infty(\Delta^k, M)$ concentrated in σ, we obtain for the right hand side

$$\langle f, i_M(c) \rangle = \int f\, d\delta_\sigma = 1 \cdot f(\sigma).$$

The corresponding equality in (co)homology follows because $B\colon \widehat{\mathcal{C}}^*(M) \longrightarrow \widehat{C}^*_{\mathrm{sm}}(M)$ is easily recognised to be a chain map. $\square$

The above Kronecker product leads to the following (slightly weakened) duality principle.

Lemma (4.23) (Duality Principle of Measure Homology). *Let M be a smooth manifold without boundary, $k \in \mathbf{N}$, and $\alpha \in \mathcal{H}_k(M)$.*

1. *If $\|\alpha\|_{\mathrm{mh}} = 0$, then for all $\varphi \in \widehat{\mathcal{H}}^k(M)$*
$$\langle \varphi, \alpha \rangle = 0.$$
2. *If $\|\alpha\|_{\mathrm{mh}} > 0$, then*
$$\|\alpha\|_{\mathrm{mh}} \geq \sup\Big\{ \frac{1}{\|\varphi\|_\infty} \Bigm| \varphi \in \widehat{\mathcal{H}}^k(M),\, \langle \varphi, \alpha \rangle = 1 \Big\}.$$

Proof. Let $\varphi \in \widehat{\mathcal{C}}^k(M)$. Assume that $\mu \in \mathcal{C}_k(M)$ is a measure cycle representing α and $f \in \widehat{\mathcal{C}}^k(M)$ is a cocycle representing φ. If $\langle \varphi, \alpha \rangle = 1$, then

$$\begin{aligned} 1 &= |\langle \varphi, \alpha \rangle| = |\langle f, \mu \rangle| \\ &= \Big| \int f\, d\mu \Big| \leq \int |f|\, d\mu_+ + \int |f|\, d\mu_- \\ &\leq \|f\|_\infty \cdot \int 1\, d\mu_+ + \|f\|_\infty \cdot \int 1\, d\mu_- \\ &= \|f\|_\infty \cdot \|\mu\|. \end{aligned}$$

Taking the infimum over all representatives results in $1 \leq \|\varphi\|_\infty \cdot \|\alpha\|_{\mathrm{mh}}$. In particular, if there exists such a φ, then

$$\|\alpha\|_{\mathrm{mh}} \geq \frac{1}{\|\varphi\|_\infty} > 0.$$

Now the lemma is an easy consequence of this inequality. $\square$

A posteriori we will be able conclude – in view of Theorem (4.14), and Lemma (4.25) – that in the first part of the lemma "if and only if" is also true and that in the second part equality holds.

4.3.2 ■ Proof of Theorem (4.14)

The dual $\widehat{C}^*(M)$ will be investigated by means of the complex $\mathsf{I}^*(M)$ introduced in Definition (2.36): the vector space $\mathsf{I}^k(M)$ is the set of all bounded functions in $\mathrm{map}(\widetilde{M}^{k+1}, \mathbf{R})$ which are $\pi_1(M)$-invariant. Then the key to the proof of Theorem (4.14) is a careful analysis of the diamond

$$\begin{tikzcd} & \mathsf{I}^*(M) \arrow[dl, "A"'] \arrow[dr, "D"] & \\ \widehat{C}^*(M) \arrow[dr, "B"'] & & \widehat{C}^*(M), \arrow[dl, "E"] \\ & \widehat{C}^*_{\mathrm{sm}}(M) & \end{tikzcd}$$

the maps being defined as follows:

- For $k \in \mathbf{N}$ let $s_k \colon \mathrm{map}(\Delta^k, M) \longrightarrow \mathrm{map}(\Delta^k, \widetilde{M})$ be a Borel section of the map induced by the universal covering map. The existence of such a section is guaranteed by the following theorem:

  Theorem (4.24). *Let M be a smooth manifold without boundary with universal covering map $p\colon \widetilde{M} \longrightarrow M$, and let $k \in \mathbf{N}$.*

  1. *Then the map*
  $$\begin{aligned} P\colon \mathrm{map}\bigl(\Delta^k, \widetilde{M}\bigr) &\longrightarrow \mathrm{map}\bigl(\Delta^k, M\bigr) \\ \sigma &\longmapsto p \circ \sigma \end{aligned}$$
  is a local homeomorphism.
  2. *Moreover, there exists a Borel section of P.*

  The (elementary, but rather technical) proof of this theorem is exiled to Subsection 4.3.3.

If $f \in \mathsf{I}^k(M)$, we write

$$\begin{aligned} A(f)\colon \operatorname{map}_\infty\bigl(\Delta^k, M\bigr) &\longrightarrow \mathbf{R} \\ \sigma &\longmapsto f\bigl((s_k(\sigma))(e_0), \dots, (s_k(\sigma))(e_k)\bigr), \end{aligned}$$

using the inclusion $\operatorname{map}_\infty(\Delta^k, M) \hookrightarrow \operatorname{map}(\Delta^k, M)$.

- Similarly (see also Section 2.5), we define $D(f)$ for $f \in \mathsf{I}^k(M)$ as the linear extension to $C_k(M, \mathbf{R})$ of the map

$$\begin{aligned} S_k(M) = \operatorname{map}\bigl(\Delta^k, M\bigr) &\longrightarrow \mathbf{R} \\ \sigma &\longmapsto f\bigl((s_k(\sigma))(e_0), \dots, (s_k(\sigma))(e_k)\bigr). \end{aligned}$$

- The map B is given by linear extension (cf. Lemma (4.22)).
- The map E is given by restriction, i.e., $E = \widehat{C}^*(j_M)$.

This diagram allows us to compare the map B (which is the building bridge between $\widehat{\mathcal{H}}^*(M)$ and $\widehat{H}^*_{\mathrm{sm}}(M)$) with D and E, which both induce isometric isomorphisms on the level of bounded cohomology.

Lemma (4.25). *With the above notation the following statements hold:*

1. *The maps A, B, D, and E are well-defined cochain maps.*
2. *Moreover, $\|A(f)\|_\infty \leq \|f\|_\infty$ for all $f \in \mathsf{I}^k(M)$ and all $k \in \mathbf{N}$. This implies*

$$\forall_{\xi \in H^k(\mathsf{I}^*(M))} \quad \bigl\|H^*(A)(\xi)\bigr\|_\infty \leq \|\xi\|_\infty .$$

3. *The diagram commutes, that is $B \circ A = E \circ D$.*
4. *The homomorphism $H^*(B)\colon \widehat{\mathcal{H}}^*(M) \longrightarrow \widehat{H}^*_{\mathrm{sm}}(M)$ induced by the cochain map B is surjective.*

Proof. 1. The fact that B and E are well-defined and compatible with the respective coboundary operators can be directly read off the definitions.

In Section 2.5 it is proved that D does not depend on the choice of s_k and that D is a cochain map. The same calculations show that the definition of A is independent of the choice of s_k and that A is a cochain map. Let

$f \in \mathsf{I}^k(M)$. Then $A(f)$ is obviously bounded. Moreover, $A(f)$ is measurable: By construction, $A(f)$ is the composition

$$\begin{array}{ccccc} \operatorname{map}_\infty(\Delta^k, M) \hookrightarrow \operatorname{map}(\Delta^k, M) & \xrightarrow{s_k} & \operatorname{map}(\Delta^k, \widetilde{M}) \to & \widetilde{M}^{k+1} & \xrightarrow{f} \mathbf{R} \\ \sigma \mapsto \sigma & & \tau \mapsto & (\tau(e_0), \dots, \tau(e_k)) & \end{array}$$

which consists of measurable and continuous functions (see Lemma (3.23), Remark (3.24), Remark (3.11)).

2. This is an immediate consequence of the definition of A.

3. For all $f \in \mathsf{I}^k(M)$ and all $\sigma \in \operatorname{map}_\infty(\Delta^k, M)$ we obtain

$$\begin{aligned} \big(B \circ A(f)\big)(\sigma) &= \big(A(f)\big)(\sigma) \\ &= f\big((s_k(\sigma))(e_0), \dots, (s_k(\sigma))(e_k)\big) \\ &= \big(D(f)\big)\big(j_M(\sigma)\big) \\ &= \big(E \circ D(f)\big)(\sigma). \end{aligned}$$

4. Due to Corollary (2.38) and Theorem (4.7), the chain maps D and E induce isomorphisms on cohomology. Since the above diagram commutes, $H^*(B)$ must be surjective. □

We have now collected all the necessary tools to prove that measure homology and (smooth) singular homology are isometrically isomorphic.

Proof (of Theorem (4.14)). According to Theorem (4.10), the induced homomorphism $H_*(i_M)\colon H_*^{\mathrm{sm}}(M, \mathbf{R}) \longrightarrow \mathcal{H}_*(M)$ is an isomorphism. Therefore, it remains to show that $H_*(i_M)$ is compatible with the seminorms.

Let $k \in \mathbf{N}$ and $\alpha \in H_k^{\mathrm{sm}}(M, \mathbf{R})$. Since $i_M\colon C_*^{\mathrm{sm}}(M, \mathbf{R}) \longrightarrow \mathcal{C}_*(M)$ is norm preserving, it is immediate that

$$\|H_k(i_M)(\alpha)\|_{\mathrm{mh}} \leq \|\alpha\|_1^{\mathrm{sm}}.$$

The proof of the reverse inequality is split into two cases:

① Suppose $\big\|H_k(i_M)(\alpha)\big\|_{\mathrm{mh}} = 0$. From Lemma (4.22) and Lemma (4.23) we obtain

$$\big\langle H^k(B)(\varphi), \alpha \big\rangle = \big\langle \varphi, H_k(i_M)(\alpha) \big\rangle = 0$$

for all $\varphi \in \widehat{\mathcal{H}}^k(M)$. By the previous lemma, $\widehat{H}^k(B)$ is surjective. Hence

$$\forall_{\psi \in \widehat{H}^k_{\mathrm{sm}}(M)} \quad \langle \psi, \alpha \rangle = 0,$$

implying $\|\alpha\|_1^{\mathrm{sm}} = 0$.

② Let $\big\|H_k(i_M)(\alpha)\big\|_{\mathrm{mh}} > 0$. In this case, Lemma (4.23) and Lemma (4.22) yield

$$\begin{aligned}\big\|H_k(i_M)(\alpha)\big\|_{\mathrm{mh}} &\geq \sup\Big\{\frac{1}{\|\varphi\|_\infty}\ \Big|\ \varphi \in \widehat{\mathcal{H}}^k(M), \big\langle\varphi, H_k(i_M)(\alpha)\big\rangle = 1\Big\}\\ &= \sup\Big\{\frac{1}{\|\varphi\|_\infty}\ \Big|\ \varphi \in \widehat{\mathcal{H}}^k(M), \big\langle H^k(B)(\varphi), \alpha\big\rangle = 1\Big\}.\end{aligned}$$

We will compare the last set with the corresponding set of Corollary (4.8): Let $\psi \in \widehat{H}^k_{\mathrm{sm}}(M)$ such that $\langle\psi, \alpha\rangle = 1$. Since the composition $H^k(E) \circ H^k(D)$ is an isometric isomorphism (Corollary (2.38) and Theorem (4.7)), there exists a $\xi \in H^k\big(\mathsf{I}^*(M)\big)$ satisfying

$$\big(H^k(E) \circ H^k(D)\big)(\xi) = \psi \quad \text{and} \quad \|\xi\|_\infty = \|\psi\|_\infty.$$

Then $\varphi := H^k(A)(\xi) \in \widehat{\mathcal{H}}^k(M)$ possesses the following properties:

- By construction,

  $$H^k(B)(\varphi) = \big(H^k(B) \circ H^k(A)\big)(\xi) = \big(H^k(E) \circ H^k(D)\big)(\xi) = \psi,$$

  and hence
  $$\big\langle H^k(B)(\varphi), \alpha\big\rangle = \langle\psi, \alpha\rangle = 1.$$

- Furthermore, we get from Lemma (4.25)

  $$\|\varphi\|_\infty = \big\|H^k(A)(\xi)\big\|_\infty \leq \|\xi\|_\infty = \|\psi\|_\infty.$$

Combining these properties with the above estimate results in

$$\begin{aligned}\big\|H_k(i_M)(\alpha)\big\|_{\mathrm{mh}} &\geq \sup\Big\{\frac{1}{\|\varphi\|_\infty}\ \Big|\ \varphi \in \widehat{\mathcal{H}}^k(M), \big\langle H^k(B)(\varphi), \alpha\big\rangle = 1\Big\}\\ &\geq \sup\Big\{\frac{1}{\|\psi\|_\infty}\ \Big|\ \psi \in \widehat{H}^k_{\mathrm{sm}}(M), \langle\psi, \alpha\rangle = 1\Big\}.\end{aligned}$$

Since $\|\alpha\|_1^{\mathrm{sm}} \geq \|H_k(i_M)(\alpha)\|_{\mathrm{mh}} > 0$, we can use Lemma (4.8) to conclude

$$\|H_k(i_M)(\alpha)\|_{\mathrm{mh}} \geq \|\alpha\|_1^{\mathrm{sm}}. \qquad \square$$

4.3.3 ■ Existence of a Borel section

To complete the proof of Theorem (4.14), we still have to provide a proof of Theorem (4.24):

Proof (of Theorem (4.24)). Since M is smooth, we may assume that $\widetilde{M}$ is also a smooth manifold and p is a local diffeomorphism. Moreover, the smooth manifold M can be equipped with a Riemannian metric. Then there is a Riemannian metric on $\widetilde{M}$ such that p is a local isometry (e.g., the pulled back Riemannian metric). The topologies on M and $\widetilde{M}$ are induced by the metrics d_M and $d_{\widetilde{M}}$ which are given by the respective Riemannian metrics. Then p is also a local isometry with respect to d_M and $d_{\widetilde{M}}$.

1. Let $\sigma \in \operatorname{map}(\Delta^k, \widetilde{M})$. Since $p\colon \widetilde{M} \longrightarrow M$ is a covering map, there is a small neighbourhood U of $\sigma(e_0)$ in $\widetilde{M}$, which means:

Definition (4.26). An open subset $U \subset \widetilde{M}$ is called **small** if $p(U) \subset M$ is open and $p|_U\colon U \longrightarrow p(U)$ is a homeomorphism. ◇

We will show that $P(U^{\{e_0\}})$ is open and that $P|_{U^{\{e_0\}}}\colon U^{\{e_0\}} \longrightarrow P(U^{\{e_0\}})$ is a homeomorphism.

The set $P(U^{\{e_0\}})$ is open in $\operatorname{map}(\Delta^k, M)$. By definition, $P(U^{\{e_0\}}) \subset \big(p(U)\big)^{\{e_0\}}$. On the other hand, for each $\tau \in \big(p(U)\big)^{\{e_0\}}$ there exists a lift $\widetilde{\tau}\colon \Delta^k \longrightarrow \widetilde{M}$ such that $\widetilde{\tau}(e_0) \in U$ since Δ^k is simply connected. Thus

$$P(U^{\{e_0\}}) = \big(p(U)\big)^{\{e_0\}}.$$

Since $p(U)$ is open in M, this is an open subset of $\operatorname{map}(\Delta^k, M)$.

The restriction $P|_{U^{\{e_0\}}}\colon U^{\{e_0\}} \longrightarrow \big(p(U)\big)^{\{e_0\}} = P\big(U^{\{e_0\}}\big)$ is bijective. Since U is small, $p|_U$ is injective. Hence the uniqueness of lifts (prescribed on e_0 by the property to map into U) shows injectivity (Δ^k is connected).

The restriction $P|_{U^{\{e_0\}}}\colon U^{\{e_0\}} \longrightarrow \big(p(U)\big)^{\{e_0\}} = P\big(U^{\{e_0\}}\big)$ is a homeomorphism. Because of Lemma (3.15), the map P is continuous. It therefore remains to prove that the restriction $P|_{U^{\{e_0\}}}$ is open:

Let $\varrho \in U^{\{e_0\}}$. Since Δ^k is compact and $p \circ \varrho$ is continuous, the set

$$K := p \circ \varrho(\Delta^k)$$

is compact.

Then the "distance" of the sheets of p over K is uniformly bounded from below in the following sense:

Lemma (4.27). *There is a constant $C \in \mathbf{R}_{>0}$ satisfying the following property: the distance of all points $y_1, y_2 \in p^{-1}(K)$ such that $p(y_1) = p(y_2)$ and $y_1 \neq y_2$ is bounded from below via*

$$d_{\widetilde{M}}(y_1, y_2) \geq C.$$

Proof (of Lemma (4.27)). We consider the function

$$\begin{aligned} D\colon K &\longrightarrow \mathbf{R} \\ x &\longmapsto \inf_{y \in p^{-1}(x) \setminus \{\widetilde{x}\}} d_{\widetilde{M}}(\widetilde{x}, y), \end{aligned}$$

where $\widetilde{x}$ is some p-lift of x. The function D is independent of this choice since the fundamental group $\pi_1(M)$ acts isometrically on $\widetilde{M}$ and transitively on each fibre (Corollary (5.10) and [Massey, Lemma V 8.1]).

Let $x \in K$. Then there is an open neighbourhood W of x such that there is a trivialisation

$$\begin{tikzcd} W \times \pi_1(M) \arrow[rr, "\varphi_W", "\cong"'] \arrow[dr, "\mathrm{pr}_W"'] & & p^{-1}(W) \arrow[dl, "p"] \\ & W & \end{tikzcd}$$

and $p|_{\varphi_W(W \times \{1\})} \colon \varphi_W(W \times \{1\}) \longrightarrow W$ is an isometric diffeomorphism.

Let $\widetilde{x} := \varphi_W(x, 1)$. Since W is an open neighbourhood of x, the image $\varphi_W(W \times \{1\})$ is an open neighbourhood of $\widetilde{x}$ in $p^{-1}(W)$ (the fibre $\pi_1(M)$ carries the discrete topology) and hence in $\widetilde{M}$. So there is an $r_x \in \mathbf{R}_{>0}$ such that the open ball $B^{\widetilde{M}}_{2r_x}(\widetilde{x})$ is contained in $\varphi_W(W \times \{1\})$ and $p\bigl(B^{\widetilde{M}}_{2r_x}(\widetilde{x})\bigr) = B^M_{2r_x}(x)$.

Let z be a point in the open neighbourhood $V_x := B^M_{r_x}(x)$ of x. We have $B^M_{r_x}(x) \subset W$ because $p|_{\varphi_W(W \times \{1\})}$ is isometric and $B^M_{2r_x}(x) = p\bigl(B^{\widetilde{M}}_{2r_x}(\widetilde{x})\bigr)$. Since φ_W is injective and $B^{\widetilde{M}}_{2r_x}(\widetilde{x}) \subset \varphi_W(W \times \{1\})$, we conclude

$$\varphi_W(z, h) \notin B^{\widetilde{M}}_{2r_x}(\widetilde{x})$$

for all $h \in \pi_1(M) \setminus \{1\}$. In particular,

$$\begin{aligned} d_{\widetilde{M}}\bigl(\varphi_W(z,h), \varphi_W(z,1)\bigr) &\geq d_{\widetilde{M}}\bigl(\varphi_W(z,h), \widetilde{x}\bigr) - d_{\widetilde{M}}\bigl(\widetilde{x}, \varphi_W(z,1)\bigr) \\ &\geq 2 \cdot r_x - d_M(x, z) \\ &\geq 2 \cdot r_x - r_x \\ &= r_x \end{aligned}$$

for all $h \in \pi_1(M) \setminus \{1\}$. This yields $D(z) \geq r_x$ for all $z \in V_x$. Since K is compact, it can be covered by finitely many such sets $V_{x_1}, \dots, V_{x_n}$. Then

$$C := \min_{j \in \{1,\dots,n\}} r_{x_j}$$

gives the desired estimate. □

We will show below that for all small enough $\varepsilon \in \mathbf{R}_{>0}$ the image of

$$\widetilde{V}_\varepsilon := \Big\{\tau \in \operatorname{map}(\Delta^k, \widetilde{M}) \;\Big|\; \sup_{x \in \Delta^k} d_{\widetilde{M}}(\tau(x), \varrho(x)) < \varepsilon\Big\}$$

under $P|_{U^{\{e_0\}}}$ is the set

$$V_\varepsilon := \Big\{\tau \in \operatorname{map}(\Delta^k, M) \;\Big|\; \sup_{x \in \Delta^k} d_M(\tau(x), p \circ \varrho(x)) < \varepsilon\Big\}.$$

As Δ^k is compact, the topologies on $\operatorname{map}(\Delta^k, \widetilde{M})$ and $\operatorname{map}(\Delta^k, M)$ are the same as the respective topologies of uniform convergence (Remark (3.13)). This proves that $P|_{U^{\{e_0\}}}$ is open.

The idea is to apply the above lemma in the following way: since the sheets of the covering are far away from each other and functions in V_ε are very close, their lifts must lie in the "same" sheet. Using the fact that p is a local isometry, we can conclude that the lifts are also very close.

In order to make the second statement precise, we need the following result:

Lemma (4.28). *There is a constant $\delta \in \mathbf{R}_{>0}$ such that for each $y \in p^{-1}(K)$ the restriction*

$$p|_{B_\delta^{\widetilde{M}}(y)} \colon B_\delta^{\widetilde{M}}(y) \longrightarrow p\big(B_\delta^{\widetilde{M}}(y)\big)$$

to the open ball of radius δ around y (with respect to $d_{\widetilde{M}}$) is an isometric diffeomorphism and $p\big(B_\delta^{\widetilde{M}}(y)\big) = B_\delta^M(p(y))$.

Proof (of Lemma (4.28)). For each $x \in K$ we choose an element $\widetilde{x} \in p^{-1}(x)$. Then there are balls $B_{\varepsilon_x}^{\widetilde{M}}(\widetilde{x})$ such that $p|_{B_{\varepsilon_x}^{\widetilde{M}}(\widetilde{x})} \colon B_{\varepsilon_x}^{\widetilde{M}}(\widetilde{x}) \longrightarrow p\big(B_{\varepsilon_x}^{\widetilde{M}}(\widetilde{x})\big)$ is an isometric diffeomorphism and $p\big(B_{\varepsilon_x}^{\widetilde{M}}(\widetilde{x})\big) = B_{\varepsilon_x}^M(x)$.

Since $\pi_1(M)$ acts by isometries on $\widetilde{M}$ (see Corollary (5.10)), we conclude that

$$g \cdot B_{\varepsilon_x}^{\widetilde{M}}(\widetilde{x}) = B_{\varepsilon_x}^{\widetilde{M}}(g \cdot \widetilde{x})$$

and that
$$p|_{g \cdot B^{\widetilde{M}}_{\varepsilon_x}(\widetilde{x})} \colon g \cdot B^{\widetilde{M}}_{\varepsilon_x}(\widetilde{x}) \longrightarrow p\big(B^{\widetilde{M}}_{\varepsilon_x}(\widetilde{x})\big)$$
is an isometric diffeomorphism for all $g \in \pi_1(M)$. As K is compact, we can find an $n \in \mathbf{N}$ and $x_1, \dots, x_n \in K$ such that
$$K \subset \bigcup_{j \in \{1,\dots,n\}} B^M_{\varepsilon_{x_j}/2}(x_j).$$
Then
$$\delta := \min_{j \in \{1,\dots,n\}} \frac{\varepsilon_{x_j}}{2}$$
is the desired constant: namely, let $y \in p^{-1}(K)$ and $x := p(y)$. Then there is a $j \in \{1, \dots, n\}$ such that $x \in B^M_{\varepsilon_{x_j}/2}(x_j)$. Since $\pi_1(M)$ acts transitively on each fibre, there is a $g \in \pi_1(M)$ satisfying $y \in g \cdot B^{\widetilde{M}}_{\varepsilon_{x_j}}(\widetilde{x}_j)$. Because $p|_{B^{\widetilde{M}}_{\varepsilon_{x_j}}(g \cdot \widetilde{x}_j)}$ is an isometry,
$$d_{\widetilde{M}}(y, g \cdot \widetilde{x}_j) = d_M(x, x_j) \le \frac{\varepsilon_{x_j}}{2}.$$
In particular, $B^{\widetilde{M}}_\delta(y) \subset B^{\widetilde{M}}_{\varepsilon_{x_j}}(g \cdot \widetilde{x}_j)$. But then
$$p|_{B^{\widetilde{M}}_\delta(y)} \colon B^{\widetilde{M}}_\delta(y) \longrightarrow p\big(B^{\widetilde{M}}_\delta(y)\big)$$
is also an isometric diffeomorphism. Moreover,
$$B^M_\delta(x) \subset B^M_{\varepsilon_{x_j}}(x_j) = p\big(B^{\widetilde{M}}_{\varepsilon_{x_j}}(\widetilde{x}_j)\big),$$
since $x \in B^M_{\varepsilon_{x_j}/2}(x_j)$. Therefore $p\big(B^{\widetilde{M}}_\delta(y)\big) = B^M_\delta(p(y))$. ☐

How small does ε have to be? We take $\varepsilon \in \big(0, \min(C/4, \delta/3)\big)$ so small that $\widetilde{V}_\varepsilon \subset U^{\{e_0\}}$, where C is the constant of Lemma (4.27) and δ is constructed in Lemma (4.28).

Since $\varepsilon < \delta$, we conclude that $P(\widetilde{V}_\varepsilon) \subset V_\varepsilon$: Let $\tau \in \widetilde{V}_\varepsilon$ and $x \in \Delta^k$. Since $\varrho(x) \in p^{-1}(K)$ and $\varepsilon < \delta$, it follows that
$$d_M\big(p \circ \tau(x), p \circ \varrho(x)\big) = d_{\widetilde{M}}\big(\tau(x), \varrho(x)\big) < \varepsilon.$$
Hence $P(\tau) = p \circ \tau \in V_\varepsilon$.

Conversely, let $\tau \in V_\varepsilon$. Since Δ^k is simply connected, there is a lift $\widetilde{\tau}$ of τ. Moreover, we may choose $\widetilde{\tau}(e_0)$ in such a way that

$$d_{\widetilde{M}}\big(\widetilde{\tau}(e_0), \varrho(e_0)\big) < \varepsilon$$

because $\varrho(e_0) \in p^{-1}(K)$ and $d_M\big(\tau(e_0), p \circ \varrho(e_0)\big) < \varepsilon < \delta$. In order to show that $\widetilde{\tau} \in \widetilde{V}_\varepsilon$ we have to check that the set

$$D := \big\{x \in \Delta^k \bigm| d_{\widetilde{M}}\big(\widetilde{\tau}(x), \varrho(x)\big) < \varepsilon\big\}$$

equals Δ^k.

By construction, $e_0 \in D$. In particular, D is not empty. Since Δ^k is connected, it suffices to prove that D is open and closed:

- *The set D is open.* Let $x \in D$. Since $\widetilde{\tau}$ and ϱ are continuous, there is an open neighbourhood W of x such that

  $$\forall_{y \in W} \quad d_{\widetilde{M}}\big(\widetilde{\tau}(y), \widetilde{\tau}(x)\big) < \varepsilon \quad \text{and} \quad d_{\widetilde{M}}\big(\varrho(y), \varrho(x)\big) < \varepsilon.$$

  Hence

  $$\begin{aligned} d_{\widetilde{M}}\big(\widetilde{\tau}(y), \varrho(y)\big) &\leq d_{\widetilde{M}}\big(\widetilde{\tau}(y), \widetilde{\tau}(x)\big) + d_{\widetilde{M}}\big(\widetilde{\tau}(x), \varrho(x)\big) + d_{\widetilde{M}}\big(\varrho(x), \varrho(y)\big) \\ &< \varepsilon + \varepsilon + \varepsilon \leq \delta \end{aligned}$$

  for all $y \in W$. Now the construction of δ yields

  $$d_{\widetilde{M}}\big(\widetilde{\tau}(y), \varrho(y)\big) = d_M\big(\tau(y), p \circ \varrho(y)\big) < \varepsilon.$$

  Thus $W \subset D$, implying that D is open.

- *The set D is closed.* Suppose $x \in \Delta^k \setminus D$, i.e., $d_{\widetilde{M}}(\widetilde{\tau}(x), \varrho(x)) \geq \varepsilon$. The condition $\varepsilon < \delta$ assures that p is isometric on the ε-ball in $\widetilde{M}$ around $\varrho(x)$ and that $p\big(B_\varepsilon^{\widetilde{M}}(\varrho(x))\big) = B_\varepsilon^M(p \circ \varrho(x))$. In particular, the ball $B_\varepsilon^{\widetilde{M}}(\varrho(x))$ contains a point y such that $p(y) = \tau(x)$ because $\tau \in V_\varepsilon$. Now Lemma (4.27) yields

  $$\begin{aligned} d_{\widetilde{M}}\big(\widetilde{\tau}(x), \varrho(x)\big) &\geq d_{\widetilde{M}}\big(\widetilde{\tau}(x), y\big) - d_{\widetilde{M}}\big(y, \varrho(x)\big) \\ &\geq C - \varepsilon. \end{aligned}$$

  By continuity, the set $W \subset \Delta^k$ of all z satisfying $d_{\widetilde{M}}(\widetilde{\tau}(z), \widetilde{\tau}(x)) < \varepsilon$ and $d_{\widetilde{M}}(\varrho(z), \varrho(x)) < \varepsilon$ is an open neighbourhood of x. Therefore, we obtain for all $z \in W$

  $$\begin{aligned} d_{\widetilde{M}}\big(\widetilde{\tau}(z), \varrho(z)\big) &\geq d_{\widetilde{M}}\big(\widetilde{\tau}(x), \varrho(x)\big) - d_{\widetilde{M}}\big(\widetilde{\tau}(z), \widetilde{\tau}(x)\big) - d_{\widetilde{M}}\big(\varrho(x), \varrho(z)\big) \\ &\geq C - \varepsilon - \varepsilon - \varepsilon \geq \varepsilon. \end{aligned}$$

  Hence $W \subset \Delta^k \setminus D$, which shows that D is closed.

This proves the first part of Theorem (4.24).

2. As for the second part note that we can cover $\operatorname{map}(\Delta^k, \widetilde{M})$ with countably many open sets $(V_n)_{n\in\mathbf{N}}$ on which P is a homeomorphism (e.g., one could take a countable covering of $\widetilde{M}$ by small sets U and consider sets of the form $U^{\{e_0\}}$). In particular, the sets $(P(V_n))_{n\in\mathbf{N}}$ are open in $\operatorname{map}(\Delta^k, M)$. Via $W_0 := P(V_0)$ and

$$\forall_{n\in\mathbf{N}} \quad W_{n+1} := P(V_{n+1}) \setminus \bigcup_{j\in\{0,\dots,n\}} W_j$$

we get a countable family $(W_n)_{n\in\mathbf{N}}$ of mutually disjoint Borel subsets in $\operatorname{map}(\Delta^k, M)$ such that $P^{-1}|_{W_n}$ is well-defined and continuous for each $n \in \mathbf{N}$. Moreover, $\operatorname{map}(\Delta^k, M)$ is covered by the $(W_n)_{n\in\mathbf{N}}$ because P is surjective. Putting all these maps together yields the desired Borel section of P. □

4.3.4 ■ Bowen's argument

In the following, flaws in Bowen's preprint "An Isometry Between Measure Homology and Singular Homology" [Bowen] are outlined, showing that his approach does not work. In this section, we will use the notation of his preprint.

Bowen's idea is to first improve measure cycles (without changing the represented class in homology) and then explicitly construct a singular chain representing the same class. Both steps are designed in such a manner that the norm is not increased.

For the first step a map $R\colon \mathcal{C}_*(M) \longrightarrow \mathcal{C}_*(M)$ is constructed, mapping measure chains to these improved chains. Bowen claims that this map R is chain homotopic to the identity. However, it is easy to see that R is not even a chain map in general.

Furthermore, the inductive construction of simplices for the second step is not possible: e.g., let σ be the 2-simplex wrapping around S^2 and mapping the boundary $\partial\Delta^2$ to the point $x_0 \in S^2$. Let τ be the 2-simplex which maps everything to x_0 (in particular, $\sigma|_{\partial\Delta^2} = \tau|_{\partial\Delta^2}$). Then $\sigma \simeq \tau$ relative to the vertices of Δ^2, but not $\sigma \simeq \tau$ relative to the boundary $\partial\Delta^2$.

This example also shows that, in general, $[j(c)] \neq [\mu]$: Let $\mu := j(\sigma - \tau)$. This is a cycle and μ generates $H_2\left(S^2, \mathbf{R}\right) \cong \mathbf{R}$. However, $\sigma \in H(\tau)$. Therefore,

$$j(c) = \mu\big(H(\sigma)\big) \cdot j(\tau) = (1-1) \cdot j(\tau) = 0.$$

In particular, $[j(c)] \neq [\mu]$.

4.4 ■ Integrating measure homology chains

As in (smooth) singular homology theory (Lemma (1.10)), we would like to introduce an integration process for measure chains to get information about the measure homology classes represented by certain cycles.

Definition (4.29). Let M be a smooth manifold without boundary. Let $k \in \mathbf{N}$, $\sigma \in \operatorname{map}_\infty(\Delta^k, M)$ and ω be a smooth k-form on M. Then there is an extension (f, V) of σ and we define

$$\int_\sigma \omega := \int_{\Delta^k} f^*\omega,$$

where $f^*\omega$ denotes the pulled back k-form on the k-manifold V. ◇

Integrating forms over smooth simplices is indeed well-defined (i.e., independent of the chosen extension) since the differentials of all extensions coincide on $T\Delta^k$ (Lemma (3.20)). Hence the pulled back forms coincide over Δ^k, implying that the integral is well-defined.

Remark (4.30). If M is an oriented Riemannian manifold without boundary of dimension n, we get for all $\sigma \in \operatorname{map}_\infty(\Delta^{n+1}, M)$ via the generalised Stokes Theorem [Lee$_2$, Theorem 16.10]

$$\int_{\partial\sigma} \mathrm{vol}_M = \int_\sigma d\,\mathrm{vol}_M$$

(where vol_M is the volume form on M). Since $d\,\mathrm{vol}_M$ is an $(n+1)$-form of the n-dimensional manifold M, it must be zero. Hence

$$\int_{\partial\sigma} \mathrm{vol}_M = 0.$$

Thus the linear extension $C_n^{\mathrm{sm}}(M, \mathbf{R}) \longrightarrow \mathbf{R}$ of integrating the volume form over a smooth singular simplex induces a homomorphism

$$\begin{aligned} H_n^{\mathrm{sm}}(M, \mathbf{R}) &\longrightarrow \mathbf{R} \\ [c] &\longmapsto \int_c \mathrm{vol}_M \end{aligned}$$

on homology. ◇

The following lemma reveals the reason for using the $\mathcal{C}^1$-topology on the mapping space $\operatorname{map}_\infty(\Delta^k, M)$ instead of the coarser compact open topology. The map described in the Lemma would *not* be continuous when working with the compact open topology (since the pulled back form considerably depends on the differential).

Lemma (4.31). *Let M be a smooth manifold without boundary, $k \in \mathbf{N}$, and let ω be some smooth k-form on M. Then the map*

$$\begin{aligned} \operatorname{map}_\infty(\Delta^k, M) &\longrightarrow \mathbf{R} \\ \sigma &\longmapsto \int_\sigma \omega \end{aligned}$$

is continuous.

Proof. This is an immediate consequence of the definition of the $\mathcal{C}^1$-topology and the compatibility of integration with certain limits [Ratcliffe, Lemma 3 in §11.5]. □

Corollary (4.32). *Let M be a smooth manifold without boundary, $k \in \mathbf{N}$, and let ω be a smooth k-form on M. For each $\mu \in \mathcal{C}_k(M)$ the map*

$$\begin{aligned} \Omega\colon \operatorname{map}_\infty(\Delta^k, M) &\longrightarrow \mathbf{R} \\ \sigma &\longmapsto \int_\sigma \omega \end{aligned}$$

is μ-integrable.

Proof. Let D be a compact determination set of μ. By the previous lemma, Ω is continuous. In particular, Ω is Borel measurable and the restriction $\Omega|_D$ is bounded.

The mapping space $\operatorname{map}(\Delta^k, M)$ is Hausdorff since M is Hausdorff. Thus $\operatorname{map}_\infty(\Delta^k, M)$ is Hausdorff by Lemma (3.23). In particular, the compact set D is closed and hence Borel. This implies that

$$\overline{\Omega} := \chi_D \cdot \Omega$$

is a bounded measurable function (where χ_D stands for the characteristic function of the Borel set D). Moreover, $\operatorname{map}_\infty(\Delta^k, M) \setminus D$ must be a μ-null set and hence also a null set with respect to μ^+ and μ^-.

Since μ has finite total variation, Lemma (3.4) implies

$$\begin{aligned} \mu^+(D) &= \quad \mu(D \cap P) < \infty, \\ \mu^-(D) &= -\mu(D \cap N) < \infty, \end{aligned}$$

where $\operatorname{map}_\infty(\Delta^k, M) = P \cup N$ is a Hahn decomposition for μ. Therefore the characteristic function χ_D is μ^+-integrable and μ^--integrable. Since Ω is almost everywhere (with respect to both μ^+ and μ^-) dominated by the integrable function $\chi_D \cdot \max_{x \in D} \Omega(x)$, it is itself integrable with respect to μ^+ and μ^-. □

Hence we can integrate forms over measure chains:

Definition (4.33). Let M be a smooth manifold without boundary, $k \in \mathbf{N}$, and let ω be some smooth k-form on M. For each measure chain $\mu \in \mathcal{C}_k(M)$ we define

$$\int_\mu \omega := \int_{\operatorname{map}_\infty(\Delta^k, M)} \int_\sigma \omega \, d\mu(\sigma).$$ ◇

In fact, this integration is a generalisation of integration over smooth simplices in the following sense:

Lemma (4.34). *Let M be a smooth manifold without boundary, $k \in \mathbf{N}$, and let ω be some smooth k-form on M. For each smooth chain $c \in C_k^{\mathrm{sm}}(M, \mathbf{R})$*

$$\int_{i_M(c)} \omega = \int_c \omega$$

holds, $i_M \colon C_*^{\mathrm{sm}}(M, \mathbf{R}) \hookrightarrow \mathcal{C}_*(M)$ being the inclusion described in Lemma (4.9).

Proof. Since integration is linear, we only need to consider smooth singular simplices $\sigma \in C_k^{\mathrm{sm}}(M, \mathbf{R})$. By definition of i_M,

$$\begin{aligned} \int_{i_M(\sigma)} \omega &= \int_{\operatorname{map}_\infty(\Delta^k, M)} \int_\tau \omega \, d\big(i_M(\sigma)\big)(\tau) \\ &= \int_\sigma \omega \cdot 1, \end{aligned}$$

which proves the lemma. □

The above compatibility statement and Lemma (1.10) can be combined to yield the following description of the top measure homology group:

Theorem (4.35). *Let M be an oriented closed connected Riemannian manifold of dimension n, and let $\mu \in \mathcal{C}_n(M)$ be a measure cycle. Then the measure homology class $[\mu]$ can be expressed as follows:*

$$[\mu] = \frac{\int_\mu \mathrm{vol}_M}{\mathrm{vol}(M)} \cdot [M]_{\mathrm{mh}} \in \mathcal{H}_n(M).$$

Proof. The transformation formula shows [Ratcliffe, Lemma 4 in §11.5]

$$\forall_{\nu \in \mathcal{C}_{n+1}(M)} \quad \int_{\partial \nu} \mathrm{vol}_M = \int_\nu d\,\mathrm{vol}_M = \int_\nu 0 = 0.$$

Thus integration of the volume form induces a homomorphism

$$\begin{aligned} \mathcal{H}_n(M) &\longrightarrow \mathbf{R} \\ [\nu] &\longmapsto \int_\nu \mathrm{vol}_M . \end{aligned}$$

Denoting the canonical inclusion $C_n^{\mathrm{sm}}(M, \mathbf{R}) \hookrightarrow \mathcal{C}_n(M)$ by i_M, we obtain the commutative diagram (compare Lemma (4.34))

$$\begin{tikzcd} H_n^{\mathrm{sm}}(M, \mathbf{R}) \arrow[r, "\int_\cdot \mathrm{vol}_M"] \arrow[d, "H_n(i_M)"'] & \mathbf{R}. \\ \mathcal{H}_n(M) \arrow[ur, "\int_\cdot \mathrm{vol}_M"'] & \end{tikzcd}$$

The vertical arrow is an isomorphism by Theorem (4.10) mapping the fundamental class $[M]_{\mathrm{sm}}$ to the measure homology fundamental class $[M]_{\mathrm{mh}}$. According to Lemma (1.10)

$$\int_c \mathrm{vol}_M = \mathrm{vol}(M)$$

holds, whenever $c \in C_n^{\mathrm{sm}}(M, \mathbf{R})$ is a representative of $[M]_{\mathrm{sm}}$. Hence

$$\int_\mu \mathrm{vol}_M = \int_c \mathrm{vol}_M = \mathrm{vol}(M)$$

for each measure cycle μ representing the measure homology fundamental class $[M]_{\mathrm{mh}}$. $\square$

5 The Proportionality Principle

The proportionality principle of simplicial volume reveals a fascinating connection between the simplicial volume and the Riemannian volume: simplicial volume and Riemannian volume are proportional if we restrict our attention to manifolds which share the same universal Riemannian covering space. Similar proportionality principles also occur in the setting of L^2-invariants (Theorem (1.41)) and in the setting of characteristic numbers of complex manifolds [Hirzebruch, Satz 3].

Both Thurston and Gromov sketched (dual) proofs for the proportionality principle of simplicial volume [Thurston, page 6.9], [Gromov$_2$, Section 2.3]. Thurston's idea is to use measure homology to compute the simplicial volume and to take advantage of the larger chain complex for the so-called smearing construction. In this chapter, a detailed version of Thurston's proof will be given, based on the results of the previous chapters.

Gromov's proof uses arguments from bounded cohomology and also depends on some averaging operation via Haar measures. Unfortunately his exposition is not very explicit about certain measurability issues.

In Section 5.1, the proportionality principle is discussed and the strategy of proof is presented. Sections 5.2 and 5.3 provide the necessary tools for Thurston's smearing technique (i.e., the study of isometry groups and appropriate measures on them). In Section 5.4, the smearing of smooth singular chains is developed. The actual proof of the proportionality principle is given in Section 5.5. The chapter ends with some applications in Section 5.6.

5.1 ■ Statement of the proportionality principle

The proportionality principle of simplicial volume reveals a fascinating connection between the simplicial volume and the Riemannian volume:

Theorem (5.1) (Proportionality Principle of Simplicial Volume). *Let M and N be oriented closed connected Riemannian manifolds with isometrically isomorphic universal Riemannian coverings. Then*

$$\frac{\|M\|}{\operatorname{vol}(M)} = \frac{\|N\|}{\operatorname{vol}(N)}.$$

When speaking of a **universal Riemannian covering map**, we mean a universal covering map which is an orientation preserving local isometry.

Without loss of generality, we may assume that the universal Riemannian covering spaces of M and N are *orientation preservingly* isometrically isomorphic. Otherwise we just reverse the orientation of M.

Thus it is clear that we even may assume that M and N have the *same* universal Riemannian covering since the composition of an orientation preserving isometry and a universal Riemannian covering map again is a universal Riemannian covering map.

For convenience, we introduce the following notation, which will be used in later sections.

Setup (5.2). Let M and N be two oriented closed connected Riemannian manifolds with common universal Riemannian covering U. The covering maps $U \longrightarrow M$ and $U \longrightarrow N$ will be denoted by p_M and p_N respectively. Furthermore, we write n for the dimension of M and N. ◇

In order to prove the proportionality principle, we are looking for a chain map $C_*(N, \mathbf{R}) \longrightarrow C_*(M, \mathbf{R})$ inducing multiplication by $\operatorname{vol}(N)/\operatorname{vol}(M)$ on the top homology group without increasing the seminorm on homology. Then the inequality

$$\frac{\operatorname{vol}(N)}{\operatorname{vol}(M)} \cdot \|M\| \leq \|N\|$$

would easily follow. Swapping the rôles of M and N would give the reverse inequality and hence prove the proportionality principle.

Integration (for smooth simplices) shows that the naive approach

$$\sigma \longmapsto p_M \circ \widetilde{\sigma}$$

($\widetilde{\sigma}$ being some p_N-lift of σ) would yield the factor $\operatorname{vol}(N)/\operatorname{vol}(M)$ if it were a chain map (which it is not).

So we may try to apply some averaging operation ("smearing") to eliminate this choice of lift to get a chain map. But since $\pi_1(N)$ is not finite in general, there is no direct way to do this. Using measure homology chains instead of singular chains, we can take advantage of the Haar measure on the locally compact group $\operatorname{Isom}^+_\infty(U)$ of orientation preserving isometries on U, which contains $\pi_1(N)$. However, to satisfy the finiteness condition for measure chains, the compact quotient $\pi_1(M) \setminus \operatorname{Isom}^+_\infty(U)$ is more appropriate. To guarantee the norm condition, the constructed measure on the quotient $\pi_1(M) \setminus \operatorname{Isom}^+_\infty(U)$ is scaled to a probability measure.

The resulting chain map

$$\operatorname{smear}_{N,M} \colon C^{\mathrm{sm}}_*(N, \mathbf{R}) \longrightarrow \mathcal{C}_*(M)$$

indeed does not increase the seminorm and induces the desired factor on the level of homology – as can be seen by integration.

The smearing technique was invented by Thurston [Thurston, page 6.8]. Thurston used it primarily to prove Theorem (1.23), which he calls "Gromov's Theorem" (and which is by Gromov described as "Thurston's Theorem"). Thurston also sketches a proof of the proportionality principle based on smearing [Thurston, page 6.9]. However, he uses the isometric isomorphism of Theorem (4.14), but gives no evidence for it (in fact, a paper is cited which never appeared in the suggested form). A more detailed discussion of smearing is given in Ratcliffe's book [Ratcliffe, §11.5].

The proportionality constant in the proportionality principle is only known in very few special cases such as hyperbolic manifolds (Theorem (1.23)) or flat manifolds (Corollary (5.26)).

5.2 ■ Isometry groups

As indicated in Section 5.1, the principal underlying tool for smearing is the measure on a quotient of the group of orientation preserving isometries on the universal covering manifold induced by the Haar measure on this isometry group. So we first examine this isometry group.

Whenever we refer to a Riemannian manifold as a metric space, it is understood that the metric is the one induced by the Riemannian metric.

Remark (5.3). Let (U, g) be a Riemannian manifold without boundary. A map $f \colon U \longrightarrow U$ is a Riemannian isometry (i.e., f is smooth and the differential Tf transforms the Riemannian metric g into g) if and only if it is a metric isometry [Helgason, Theorem I 11.1]. ◇

Definition (5.4). Let U be a Riemannian manifold (without boundary, but not necessarily compact).

- The **group of (metric) isometries on** U, written as $\mathrm{Isom}_{\mathrm{co}}(U)$, is the set of all metric isometries $U \longrightarrow U$ with the multiplication given by composition, endowed with the subspace topology induced by the compact open topology on $\mathrm{map}\,(U, U)$.
- If U is oriented, the **group of orientation preserving isometries on** U, denoted by $\mathrm{Isom}^+_\infty(U)$, is the mapping space consisting of all orientation preserving isometries $U \longrightarrow U$ with the multiplication given by composition. The topology on $\mathrm{Isom}^+_\infty(U)$ is the subspace topology induced by the $\mathcal{C}^1$-topology on $\mathrm{map}_\infty\,(U, U)$. ◇

From the topologist's point of view it is of course much more comfortable to work with the group $\mathrm{Isom}_{\mathrm{co}}(U)$. However, we are forced to use the finer $\mathcal{C}^1$-topology on the space of smooth singular simplices to make the integration process in Lemma (4.31) continuous. Therefore – as can be seen in Lemma (5.16) – we also have to take the $\mathcal{C}^1$-topology on the isometry group. We will see a posteriori (Theorem (5.12)) that the compact open topology and the $\mathcal{C}^1$-topology on the isometry groups coincide. But there does not seem to exist a short way to prove this fact.

Before working out basic properties of $\mathrm{Isom}^+_\infty(U)$, we collect some facts about the coarser group $\mathrm{Isom}_{\mathrm{co}}(U)$. Lemma (5.7) will be the key for applying these results to $\mathrm{Isom}^+_\infty(U)$.

Theorem (5.5). *Let U be a Riemannian manifold.*

1. *The group* $\mathrm{Isom}_{\mathrm{co}}(U)$ *is indeed a* topological *group with respect to the given topology.*
2. *The set* $\mathrm{Isom}_{\mathrm{co}}(U)$ *is closed in* $\mathrm{map}\,(U,U)$.
3. *The group* $\mathrm{Isom}_{\mathrm{co}}(U)$ *is Hausdorff, locally compact, and satisfies the second countability axiom.*
4. *If G is a group acting properly and cocompactly on U by isometries, then G is a discrete and cocompact subgroup of* $\mathrm{Isom}_{\mathrm{co}}(U)$.

Proof. 1. A proof is given by, e.g., Helgason [Helgason, Theorem IV 2.5].

2. If $f \in \mathrm{map}\,(U,U) \setminus \mathrm{Isom}_{\mathrm{co}}(U)$, there are $x, y \in U$ such that

$$d\big(f(x), f(y)\big) \neq d(x,y),$$

where d denotes the metric on U determined by the Riemannian metric. In particular, $x \neq y$. Since $d\colon U \times U \longrightarrow \mathbf{R}_{\geq 0}$ is continuous, there are open neighbourhoods V and W of $f(x)$ and $f(y)$ respectively such that

$$\forall_{v\in V} \quad \forall_{w \in W} \quad d(v,w) \neq d(x,y).$$

Furthermore, $\{x\}$ and $\{y\}$ are obviously compact. Hence $V^{\{x\}} \cap W^{\{y\}}$ is an open neighbourhood of f in $\mathrm{map}\,(U,U)$ satisfying

$$\big(V^{\{x\}} \cap W^{\{y\}}\big) \cap \mathrm{Isom}_{\mathrm{co}}(U) = \emptyset,$$

by construction of V and W.

Parts 3. and 4. follow since Riemannian manifolds are second countable proper metric spaces [Sauer, Theorem 2.35]. The cited proof is based on the Arzela-Ascoli Theorem, and for this to apply, it is crucial that $\mathrm{Isom}_{\mathrm{co}}(U)$ is closed in $\mathrm{map}\,(U,U)$. ☐

Since the actual object of interest is the group $\mathrm{Isom}^+_\infty(U)$, we would like to have a similar theorem for $\mathrm{Isom}^+_\infty(U)$. This will be accomplished by equipping the tangential space of U with a Riemannian structure in a canonical way:

Remark (5.6). Let U be a Riemannian manifold. On the tangent bundle TU, a Riemannian metric can be defined as follows [do Carmo, Chapter 3, Exercise 2]:

Let $(p,q) \in TU$ and $v, w \in T_{(p,q)}(TU)$. Choose smooth curves

$$\begin{aligned} \alpha_v &= (p_v, q_v)\ : [-1,1] \longrightarrow TU, \\ \alpha_w &= (p_w, q_w) \colon [-1,1] \longrightarrow TU \end{aligned}$$

representing v and w respectively, i.e., $p_v(0) = p = p_w(0)$, $q_v(0) = q = q_w(0)$ and $v = \alpha_v'(0)$, $w = \alpha_w'(0)$. Then the inner product of v and w is given by

$$\langle v, w \rangle_{(p,q)} := \big\langle T_{(p,q)}\pi(v), T_{(p,q)}\pi(w) \big\rangle_p + \langle D_0 q_v, D_0 q_w \rangle_p$$

where $\pi \colon TU \longrightarrow U$ is the bundle projection, $D_0 q_v$ denotes the covariant derivative of the vector field q_v along p_v (with respect to the Riemannian connection, of course), and analogously for $D_0 q_w$. $\diamond$

Lemma (5.7). *Let U be an oriented Riemannian manifold and $f \in \mathrm{Isom}_\infty^+(U)$. Then*

$$Tf \in \mathrm{Isom}_{\mathrm{co}}(TU).$$

Proof. The construction of the smooth structure on the tangent bundle TU shows that $Tf \colon TU \longrightarrow TU$ is smooth. Therefore it remains to show that Tf preserves the Riemannian metric on TU:

Suppose $(p,q) \in TU$ and $v, w \in T_{(p,q)}(TU)$. Choose smooth curves

$$\begin{aligned} (p_v, q_v) &\colon [-1,1] \longrightarrow TU, \\ (p_w, q_w) &\colon [-1,1] \longrightarrow TU \end{aligned}$$

representing v and w respectively. Using the chain rule and the fact that $\pi \circ Tf = f \circ \pi$, we conclude

$$T_{(f(p), T_p f(q))}\pi\big(T_{(p,q)}Tf(v)\big) = T_p f\big(T_{(p,q)}\pi(v)\big).$$

Since f is an isometry of the Riemannian manifold U, this yields

$$\begin{aligned} &\big\langle T_{(f(p),T_pf(q))}\pi\big(T_{(p,q)}Tf(v)\big), T_{(f(p),T_pf(q))}\pi\big(T_{(p,q)}Tf(w)\big)\big\rangle_{f(p)} \\ &\quad = \big\langle T_p f\big(T_{(p,q)}\pi(v)\big), T_p f\big(T_{(p,q)}\pi(w)\big)\big\rangle_{f(p)} \\ &\quad = \langle T_{(p,q)}\pi(v), T_{(p,q)}\pi(w)\rangle_p. \end{aligned}$$

Considering the second summand in the definition of $\langle v, w\rangle_{(p,q)}$, we observe that (by naturality of the Riemannian connection under isometries [Lee$_1$, Proposition (5.6)])

$$D_0(Tf \circ q_v) = Tf(D_0 q_v)$$

which implies (since f is an isometry)

$$\begin{aligned}\big\langle D_0(Tf \circ q_v), D_0(Tf \circ q_w)\big\rangle_{f(p)} &= \big\langle Tf(D_0 q_v), Tf(D_0 q_w)\big\rangle_{f(p)} \\ &= \langle D_0 q_v, D_0 q_w\rangle_p.\end{aligned}$$

Thus the Riemannian metric on TU is preserved by the map Tf, i.e., Tf is an isometry of TU. □

In view of Lemma (5.7) we can use the results and methods of Theorem (5.5) to derive a corresponding theorem for the group $\mathrm{Isom}^+_\infty(U)$:

Theorem (5.8). *Let U be an oriented Riemannian manifold.*

1. *The group* $\mathrm{Isom}^+_\infty(U)$ *is indeed a* topological *group with respect to the topology from Definition (5.4).*
2. *The group* $\mathrm{Isom}^+_\infty(U)$ *is Hausdorff and satisfies the second countability axiom.*
3. *The set* $T\big(\mathrm{Isom}^+_\infty(U)\big)$ *is closed in* $\mathrm{map}_\infty(TU, TU)$.
4. *Moreover, the group* $\mathrm{Isom}^+_\infty(U)$ *is locally compact.*
5. *If G is a group acting properly and cocompactly on U by isometries, then G is a discrete and cocompact subgroup of* $\mathrm{Isom}^+_\infty(U)$.

Proof. Lemma (5.7) shows that $T\big(\mathrm{Isom}^+_\infty(U)\big) \subset \mathrm{Isom}_{\mathrm{co}}(TU)$. Therefore

$$T\colon \mathrm{Isom}^+_\infty(U) \longrightarrow T\big(\mathrm{Isom}^+_\infty(U)\big) \subset \mathrm{Isom}_{\mathrm{co}}(TU)$$

is (by definition of the topology on $\mathrm{Isom}^+_\infty(U)$) a homeomorphism.

1. Since multiplication and inverting elements on $\mathrm{Isom}_{\mathrm{co}}(TU)$ are continuous according to Theorem (5.5), this is also true for the restrictions to the subset $T\big(\mathrm{Isom}^+_\infty(U)\big)$. Thus $\mathrm{Isom}^+_\infty(U)$ is a topological group.

2. Since $\mathrm{Isom}_{\mathrm{co}}(TU)$ is Hausdorff by Theorem (5.5), the same is true for the subset $T\big(\mathrm{Isom}^+_\infty(U)\big)$. Therefore $\mathrm{Isom}^+_\infty(U)$ is also Hausdorff. By the same argument, $\mathrm{Isom}^+_\infty(U)$ is second countable.

3. Using the fact that $\mathrm{Isom}_{\mathrm{co}}(TU)$ is Hausdorff and second countable, we can check the closedness of $T\bigl(\mathrm{Isom}^+_\infty(U)\bigr)$ via sequences:

Let $(g_n)_{n\in\mathbf{N}} \subset T\bigl(\mathrm{Isom}^+_\infty(U)\bigr)$ be a sequence converging to $g \in \mathrm{Isom}_{\mathrm{co}}(TU)$. Hence we can write

$$\forall_{n\in\mathbf{N}} \quad g_n = Tf_n$$

for an appropriate sequence $(f_n)_{n\in\mathbf{N}} \subset \mathrm{Isom}^+_\infty(U)$. The convergence of the sequence $(Tf_n)_{n\in\mathbf{N}}$ to g implies that $(f_n)_{n\in\mathbf{N}}$ converges to

$$\overline{g} := \pi_U \circ g \circ i_U$$

in the compact open topology (where $i_U \colon U \longrightarrow TU$ is the inclusion of the zero-section and $\pi_U \colon TU \longrightarrow U$ denotes the bundle projection). Since i_U and π_U are smooth, it follows that $\overline{g}$ is smooth. Theorem (5.5) implies thus that $\overline{g}$ is an isometry.

To convince ourselves that $T\overline{g} = g$, we use (special) charts of TU to reduce the problem to a similar one in Euclidean space (compare Lemma (5.9)), where it can easily be solved.

Let (V_1, φ_1) be a relatively compact chart around $x \in U$ (i.e., the closure of V_1 is compact) and let (V_2, φ_2) be a chart around $\overline{g}(x) \in U$ such that $\overline{g}(V_1) \subset V_2$. Since each of the $g_n = Tf_n$ preserves fibres, the same must be true for their limit g. Then

$$g\bigl(\pi_U^{-1}(V_1)\bigr) \subset \pi_U^{-1}(V_2).$$

By definition of the manifold structure on TU, the pairs $\bigl(\pi_U^{-1}(V_1), T\varphi_1\bigr)$ and $\bigl(\pi_U^{-1}(V_2), T\varphi_2\bigr)$ are charts of TU around each point in $\pi_U^{-1}(x)$ and $\pi_U^{-1}(\overline{g}(x))$ respectively.

Since V_1 is relatively compact and V_2 is open, $f_n(V_1) \subset V_2$ holds for all large enough $n \in \mathbf{N}$. So, without loss of generality, we may assume that this is true for all $n \in \mathbf{N}$. This implies

$$Tf_n\bigl(\pi_U^{-1}(V_1)\bigr) \subset \pi_U^{-1}(V_2)$$

for all $n \in \mathbf{N}$ since Tf_n is fibre-preserving. Thus we may define

$$\begin{aligned} G &:= T\varphi_2 \circ g \circ (T\varphi_1)^{-1}, \\ F_n &:= T\varphi_2 \circ Tf_n \circ (T\varphi_1)^{-1}. \end{aligned}$$

In particular, we get commutative diagrams of the form (with $d := \dim(U)$)

$$\begin{tikzcd}
\pi_U^{-1}(V_1) \arrow[r, "g_n| = Tf_n|"] \arrow[d, "T\varphi_1"'] & \pi_U^{-1}(V_2) \arrow[d, "T\varphi_2"] \\
\varphi_1(V_1) \times \mathbf{R}^d \arrow[r, "F_n"] & \varphi_2(V_2) \times \mathbf{R}^d.
\end{tikzcd}$$

On the other hand, the differential Tf_n is characterised by the fact that the above diagram commutes when the map on the bottom is substituted by

$$\begin{aligned}
\varphi_1(V_1) \times \mathbf{R}^d &\longrightarrow \varphi_2(V_2) \times \mathbf{R}^d \\
(y, z) &\longmapsto \big((\varphi_2 \circ f_n \circ \varphi_1^{-1})(y), J(\varphi_2 \circ f_n \circ \varphi_1^{-1})_y(z)\big)
\end{aligned}$$

(here J stands for the Jacobian matrix). Hence F_n must be this (smooth) map (the vertical arrows are diffeomorphisms). Moreover, $(F_n)_{n\in\mathbf{N}}$ converges to G in the compact open topology since $T\varphi_1$ and $T\varphi_2$ are homeomorphisms.

Thus we can apply the following lemma (whose proof is deferred until the end of the proof of the theorem):

Lemma (5.9). *Let $d \in \mathbf{N}$, let V_1, $V_2 \subset \mathbf{R}^d$ be two open sets, and $(f_n)_{n\in\mathbf{N}} \subset \operatorname{map}_\infty(V_1, V_2)$. Assume that the maps $(F_n)_{n\in\mathbf{N}} \subset \operatorname{map}_\infty\big(V_1 \times \mathbf{R}^d, V_2 \times \mathbf{R}^d\big)$, defined by*

$$\begin{aligned}
F_n \colon V_1 \times \mathbf{R}^d &\longrightarrow V_2 \times \mathbf{R}^d \\
(y, z) &\longmapsto \big(f_n(y), (Jf_n)_y(z)\big)
\end{aligned}$$

converge in the compact open topology to a function $G \colon V_1 \times \mathbf{R}^d \longrightarrow V_2 \times \mathbf{R}^d$. Then $G_1 \colon V_1 \longrightarrow V_2$ is continuously differentiable and

$$\forall_{(y,z)\in V_1\times\mathbf{R}^d} \quad (JG_1)_y(z) = G_2(y, z),$$

where (denoting the obvious inclusions/projections by i_1, p_1, p_2)

$$\begin{aligned}
G_1 &:= p_1 \circ G \circ i_1, \\
G_2 &:= p_2 \circ G.
\end{aligned}$$

Since the image of the zero-section is closed in TU and preserved by all the $g_n = Tf_n$, the same must hold for g. Then a straightforward computation shows

$$\varphi_2 \circ \overline{g} \circ (\varphi_1)^{-1} = p_1 \circ G \circ i_1 =: G_1.$$

Therefore the lemma shows

$$\forall_{(y,z)\in\varphi_1(V_1)\times\mathbf{R}^d} \quad (J\overline{g})_y(z) = p_2 \circ G(y,z) =: G_2(y,z),$$

yielding the commutative diagram

$$\begin{tikzcd}
\pi_U^{-1}(V_1) \arrow[r, "g|"] \arrow[d, "T\varphi_1"'] & \pi_U^{-1}(V_2) \arrow[d, "T\varphi_2"] \\
\varphi_1(V_1)\times\mathbf{R}^d \arrow[r, "(G_1, JG_1)"] & \varphi_2(V_2)\times\mathbf{R}^d.
\end{tikzcd}$$

On the other hand, we already know that $\overline{g}$ is smooth. So the above diagram proves

$$g|_{\pi_U^{-1}(V_1)} = T\overline{g}|_{\pi_U^{-1}(V_1)}.$$

Moreover, the map $\overline{g}$ is orientation preserving: A local diffeomorphism between oriented smooth manifolds is orientation preserving if and only if its Jacobian matrix with respect to any oriented smooth charts has positive determinant [Lee$_2$, Exercise 13.5]. Using the same argument as above, we get that the Jacobians of the $(f_n)_{n\in\mathbf{N}}$ converge pointwise to the Jacobian of $\overline{g}$ with respect to such charts. Since the determinant is continuous and each f_n preserves the orientation, their limit $\overline{g}$ must also be orientation preserving.

4. To see that $\mathrm{Isom}^+_\infty(U)$ is locally compact, we apply Theorem (5.5) to the group $\mathrm{Isom}_{\mathrm{co}}(TU)$. Thus for each map $f \in \mathrm{Isom}^+_\infty(U)$ the differential $Tf \in \mathrm{Isom}_{\mathrm{co}}(TU)$ possesses a compact neighbourhood V. Then $V \cap T\bigl(\mathrm{Isom}^+_\infty(U)\bigr)$ is a compact neighbourhood of Tf in $T\bigl(\mathrm{Isom}^+_\infty(U)\bigr)$ because $T\bigl(\mathrm{Isom}^+_\infty(U)\bigr)$ is a closed subset of the Hausdorff space $\mathrm{Isom}_{\mathrm{co}}(TU)$. Therefore

$$T^{-1}\Bigl(V \cap T\bigl(\mathrm{Isom}^+_\infty(U)\bigr)\Bigr)$$

is a compact neighbourhood of f in $\mathrm{Isom}^+_\infty(U)$.

5. Due to Theorem (5.5), we know that G is a discrete subgroup of $\mathrm{Isom}_{\mathrm{co}}(U)$. Since the topology on $\mathrm{Isom}^+_\infty(U)$ is finer than the compact open topology, it is clear that $G \subset \mathrm{Isom}^+_\infty(U)$ is discrete.

We now proceed as in Sauer's proof of Theorem (5.5) [Sauer, Theorem 2.35]: Choose a point $x \in U$. Since G acts cocompactly on U, we can find a compact set $K \subset U$ such that [Elstrodt, Lemma VIII 3.19]

$$G \cdot K = M.$$

Hence for every isometry $f \in \mathrm{Isom}^+_\infty(U)$ there exists an element $g_f \in G$ such that $g_f \circ f(x) \in K$. We consider the set

$$K_G := \{T(g_f \circ f) \mid f \in \mathrm{Isom}^+_\infty(U)\}.$$

According to Lemma (5.7), we know that all the elements of K_G have Lipschitz constant 1. Moreover,

$$\begin{aligned} K_G(0_x) &:= \{T(g_f \circ f)(0_x) \mid f \in \mathrm{Isom}^+_\infty(U)\} \\ &\subset \{0_y \mid y \in \pi^{-1}(K)\} \\ &= i(K), \end{aligned}$$

where $\pi\colon TU \longrightarrow U$ denotes the bundle projection, 0_y (for $y \in U$) is the zero element of the tangent space T_yU and $i\colon U \longrightarrow TU$ is the zero section. Thus $K_G(0_x)$ is relatively compact.

Therefore, we can apply the Arzela-Ascoli Theorem [Sauer, Lemma 2.13], which shows that K_G is relatively compact in $\mathrm{map}(TU, TU)$. Since the image $T(\mathrm{Isom}^+_\infty(U))$ is a closed subset of the Hausdorff space $\mathrm{map}_\infty(TU, TU)$, it follows that $K_G \cap T(\mathrm{Isom}^+_\infty(U))$ is relatively compact in $T(\mathrm{Isom}^+_\infty(U))$. Hence

$$T|^{-1}_{\mathrm{Isom}^+_\infty(U)}(K_G) = \{g_f \circ f \mid f \in \mathrm{Isom}^+_\infty(U)\}$$

must be relatively compact in $\mathrm{Isom}^+_\infty(U)$.

Denoting the canonical projection $\mathrm{Isom}^+_\infty(U) \longrightarrow G \setminus \mathrm{Isom}^+_\infty(U)$ by π_G, we get by construction that the restriction

$$\pi_G|_{T|^{-1}_{\mathrm{Isom}^+_\infty(U)}(K_G)}\colon T|^{-1}_{\mathrm{Isom}^+_\infty(U)}(K_G) \longrightarrow G \setminus \mathrm{Isom}^+_\infty(U)$$

is surjective, implying the compactness of the quotient $G \setminus \mathrm{Isom}^+_\infty(U)$. ☐

To complete the proof of the previous theorem, it remains show Lemma (5.9):

Proof (of Lemma (5.9)). Convergence in the compact open topology coincides with uniform convergence on compact subsets (Remark (3.13)). In particular, the f_n converge pointwise to G_1 for $n \to \infty$.

Since each F_n preserves fibres and is linear in each fibre, the same holds for their limit G. If $K \subset V_1$ is compact, the product $K \times S^{d-1}$ is a compact subset of $V_1 \times \mathbf{R}^d$. Hence the F_n converge uniformly on $K \times S^{d-1}$ to G for $n \to \infty$. This implies that

$$Jf_n \longrightarrow G_2$$

uniformly on K for $n \to \infty$ (with respect to the operator norm).

A standard theorem from calculus [Lee$_2$, Theorem A.70] now shows that G_1 is continuously differentiable and that

$$\forall_{(y,z) \in V_1 \times \mathbf{R}^d} \quad (JG_1)_y(z) = G_2(y,z). \qquad \square$$

Corollary (5.10). *Let M be an oriented Riemannian manifold without boundary and U its Riemannian universal covering. Then $\pi_1(M)$ can be considered a subgroup of* $\operatorname{Isom}^+_\infty(U)$ *and the quotient $\pi_1(M) \setminus \operatorname{Isom}^+_\infty(U)$ is compact.*

Proof. Covering theory shows that $\pi_1(M)$ acts on U from the right by (mutually distinct) homeomorphisms. Since the covering map $\pi_M \colon U \longrightarrow M$ is an orientation preserving local isometry and

$$\begin{tikzcd} U \arrow[r, "\cdot\alpha"] \arrow[d, "\pi_M"'] & U \arrow[d, "\pi_M"'] \\ M \arrow[r, "\mathrm{id}"'] & M \end{tikzcd}$$

is commutative for each $\alpha \in \pi_1(M)$, it follows that $\pi_1(M)$ acts on U by orientation preserving isometries. Moreover, multiplication in $\pi_1(M)$ corresponds to composition in $\operatorname{Isom}^+_\infty(U)$ [Massey, Corollary V 7.5]. Hence $\pi_1(M)$ is a subgroup of $\operatorname{Isom}^+_\infty(U)$.

The quotient $\pi_1(M) \setminus \operatorname{Isom}^+_\infty(U)$ is compact: By covering theory, $\pi_1(M)$ acts properly on U [Massey, page 136] and $\pi_1(M) \setminus U \cong M$ is compact [Massey, Lemma 8.1 in Chapter V]. Thus $\pi_1(M) \setminus \operatorname{Isom}^+_\infty(U)$ is compact by the last part of Theorem (5.8). $\square$

Using the results of Theorem (5.8), we will now be able to conclude that the compact open and the $\mathcal{C}^1$-topology on the isometry groups coincide (but we will not need this fact in the sequel). Unfortunately, I was not able to figure out a proof of this fact which avoids the nasty computations in the proof of Theorem (5.8).

Definition (5.11). For a Riemannian manifold U, we write $\operatorname{Isom}_\infty(U)$for the group of isometries endowed with the subspace topology induced by the $\mathcal{C}^1$-topology on $\operatorname{map}_\infty(U,U)$. $\diamond$

Theorem (5.12). *Let U be a Riemannian manifold. Then the topologies on the groups* $\operatorname{Isom}_{\mathrm{co}}(U)$ *and* $\operatorname{Isom}_\infty(U)$ *are the same.*

Proof. Exactly the same methods as in the proof of Theorem (5.8) show that $\mathrm{Isom}_\infty(U)$ is a topological group which is Hausdorff, locally compact, and second countable.

We will now make use of the methods described in Kechris' book "Classical Descriptive Set Theory" [Kechris]. One of the central notions is that of a standard Borel space, which provides a convenient setting for measure theory.

As locally compact second countable Hausdorff groups, both $\mathrm{Isom}_{\mathrm{co}}(U)$ and $\mathrm{Isom}_\infty(U)$ are standard Borel groups [Kechris, Theorem 5.3]. Since the $\mathcal{C}^1$-topology is finer than the compact open topology, the identity map $\mathrm{id}\colon \mathrm{Isom}_\infty(U) \longrightarrow \mathrm{Isom}_{\mathrm{co}}(U)$ is a bijective Borel map. Then id already is a Borel isomorphism [Kechris, Theorem 14.12]. Hence $\mathrm{Isom}_\infty(U)$ and $\mathrm{Isom}_{\mathrm{co}}(U)$ coincide as Borel spaces.

This suffices to conclude that the topologies on $\mathrm{Isom}_\infty(U)$ and $\mathrm{Isom}_{\mathrm{co}}(U)$ are the same [Kechris, Proposition 12.25]. $\square$

5.3 ■ The Haar measure

In the following, some basic properties of Haar measures are recalled. Haar measures are regular measures on locally compact groups which are compatible with the group structure. Using the results from the previous section, we will prove that there is a right invariant probability measure on the Borel σ-algebra of the quotient $\pi_1(M) \setminus \mathrm{Isom}^+_\infty(U)$. This measure will be used in the averaging process mentioned in the introduction.

Definition (5.13). Let G be a locally compact Hausdorff topological group, and let $\mathcal{A}$ be its Borel σ-algebra.

- A (positive) measure μ on $\mathcal{A}$ is called **regular** if

$$\mu(A) = \sup\{\mu(K) \mid K \text{ compact}, K \subset A\},$$
$$\mu(A) = \inf\{\mu(U) \mid U \text{ open}, A \subset U\}$$

  holds for all $A \in \mathcal{A}$, and if $\mu(K)$ is finite for all compact sets $K \subset G$.

- A **Haar measure on** G is a positive measure h on $\mathcal{A}$ which is regular, non-zero on any non-empty open set, and which is **left invariant**, i.e.,

$$h(g \cdot A) = h(A)$$

  for all $g \in G$ and all $A \in \mathcal{A}$.

- The group G is called **unimodular** if each left invariant Haar measure on G also is right invariant. $\diamond$

It is well-known that such a Haar measure exists for each locally compact Hausdorff group and is essentially unique (up to multiplication with a positive constant) [Elstrodt, Satz VIII 3.12].

For example, the Haar measure on the discrete group $\mathbf{Z}$ is the counting measure, and the Haar measure on the additive group $\mathbf{R}$ is the Lebesgue measure. Obviously, all Abelian groups are unimodular. Additionally, all compact groups are unimodular [Elstrodt, Satz VIII 3.16], and – as we will show – the isometry group $\operatorname{Isom}^+_\infty(U)$ of an oriented Riemannian manifold is unimodular.

Definition (5.14). Let X be a topological space, let μ be a measure on the Borel σ-algebra of X, and suppose that the group G acts via Borel isomorphisms on X (from the left). A **measure fundamental domain** of this action is a Borel set $F \subset X$ such that

$$\mu(X \setminus G \cdot F) = 0,$$

and

$$\forall_{g \in G \setminus \{1\}} \quad \mu(g \cdot F \cap F) = 0. \qquad \diamond$$

Theorem (5.15). *Presuppose the notation from Setup (5.2). There exists a (positive) right invariant measure h_M on the Borel σ-algebra of $\pi_1(M) \setminus \operatorname{Isom}^+_\infty(U)$ satisfying*

$$h_M\bigl(\pi_1(M) \setminus \operatorname{Isom}^+_\infty(U)\bigr) = 1.$$

Proof. Using properties of standard Borel spaces, it can be shown that each discrete cocompact subgroup of a locally compact second countable Hausdorff group G has **finite covolume** with respect to the Haar measure (i.e., there is a measure fundamental domain of finite measure for this subgroup) [Sauer, page 42]. Hence G must be unimodular [Sauer, Lemma 2.32].

Since $\mathrm{Isom}^+_\infty(U)$ is a locally compact second countable Hausdorff topological group (Theorem (5.8)) and $\pi_1(M) \subset \mathrm{Isom}^+_\infty(U)$ is discrete and acts cocompactly on $\mathrm{Isom}^+_\infty(U)$ (Corollary (5.10)), we conclude that $\pi_1(M)$ has finite covolume in $\mathrm{Isom}^+_\infty(U)$ and that $\mathrm{Isom}^+_\infty(U)$ is unimodular.

It follows that the Haar measure h on $\mathrm{Isom}^+_\infty(U)$ descends to a right invariant measure h' on the quotient $\pi_1(M) \setminus \mathrm{Isom}^+_\infty(U)$ [Elstrodt, Korollar 3.25 in Kapitel VIII]. If F is a finite measure fundamental domain of $\pi_1(M)$ and $\pi_M \colon \mathrm{Isom}^+_\infty(U) \longrightarrow \pi_1(M) \setminus \mathrm{Isom}^+_\infty(U)$ is the projection, then one can show that

$$h'(A) = h\big(\pi_M^{-1}(A) \cap F\big)$$

holds for all Borel subsets $A \subset \pi_1(M) \setminus \mathrm{Isom}^+_\infty(U)$:

Indeed, for each $g \in \pi_1(M)$ the translate $F \cdot g$ is also a measure fundamental domain of finite measure because h is right invariant. Left invariance of h yields that $h\big(\pi_M^{-1}(\,\cdot\,) \cap F\big)$ is independent of the chosen fundamental domain F. Then it is easy to see that $h\big(\pi_M^{-1}(\,\cdot\,) \cap F\big)$ is a right invariant measure on the quotient $\pi_1(M) \setminus \mathrm{Isom}^+_\infty(U)$.

Since $\pi_1(M)$ has finite covolume in $\mathrm{Isom}^+_\infty(U)$, it follows that $h'\big(\pi_1(M) \setminus \mathrm{Isom}^+_\infty(U)\big)$ is finite. Why is $h'\big(\pi_1(M) \setminus \mathrm{Isom}^+_\infty(U)\big) \neq 0$? Since $\pi_1(M)$ is countable and F is a measure fundamental domain, we get (where the number of non-zero summands might be infinite)

$$\begin{aligned} h\big(\mathrm{Isom}^+_\infty(U)\big) &= \sum_{g \in \pi_1(M)} h(g \cdot F) \\ &= \sum_{g \in \pi_1(M)} h(F). \end{aligned}$$

In addition, $h\big(\mathrm{Isom}^+_\infty(U)\big) \neq 0$ since Haar measures are by definition nontrivial. Thus

$$0 \neq h(F) = h'\big(\pi_1(M) \setminus \mathrm{Isom}^+_\infty(U)\big).$$

So

$$h_M := \frac{h'}{h'\big(\pi_1(M) \setminus \mathrm{Isom}^+_\infty(U)\big)}$$

is a measure with the desired properties. □

5.4 ■ Smearing

Using the methods of the preceding sections, we are now able to introduce the main construction – the "smearing" of smooth singular chains.

In this section we will always presuppose the notation from Setup (5.2). As a first step, we define for each smooth simplex σ on U a measure $\operatorname{smear}_M(\sigma)$ which is supported on all $p_M \circ \tau$, where τ is an (orientation preserving) isometric translate of σ and which is uniformly distributed – the simplex σ is "smeared" over the set $p_M \circ \bigl(\operatorname{Isom}^+_\infty(U) \cdot \sigma\bigr)$.

Subsequently, we will pull back this construction to smooth singular simplices on N. Using the results from Section 4.4, we will show that smearing induces indeed the desired factor $\operatorname{vol}(N) / \operatorname{vol}(M)$ on the top homology. Moreover, it will be fundamental that smearing does not increase the norm.

Lemma (5.16). *Let $k \in \mathbf{N}$ and $\sigma \in \operatorname{map}_\infty(\Delta^k, U)$. The map*

$$\begin{aligned} F_\sigma \colon \pi_1(M) \setminus \operatorname{Isom}^+_\infty(U) &\longrightarrow \operatorname{map}_\infty(\Delta^k, M) \\ \bigl(\pi_1(M)\bigr) \cdot f &\longmapsto p_M \circ f \circ \sigma \end{aligned}$$

is well-defined and continuous.

Since we take the $\mathcal{C}^1$-topology on the right hand side, we are forced to consider the $\mathcal{C}^1$-topology on the isometry group $\operatorname{Isom}^+_\infty(U)$ to get a continuous map.

Proof. For all $\alpha \in \pi_1(M)$ we get

$$p_M \circ (\cdot\alpha) = p_M$$

since $\pi_1(M)$ acts by deck transformations, implying that F_σ is independent of the chosen representative.

By Lemma (3.15) and Remark (3.24) the map

$$\begin{aligned} \operatorname{Isom}^+_\infty(U) &\longrightarrow \operatorname{map}_\infty(\Delta^k, M) \\ f &\longmapsto p_M \circ f \circ \sigma \end{aligned}$$

is continuous. Thus the universal property of the quotient topology yields the continuity of F_σ. □

Definition (5.17). For each $k \in \mathbf{N}$ and each $\sigma \in \operatorname{map}_\infty(\Delta^k, U)$, we define the **smeared chain of** σ by
$$\operatorname{smear}_M(\sigma) := h_M^{F_\sigma},$$
where h_M is the measure from Theorem (5.15). We extend smear_M linearly to the whole chain group $C_k^{\mathrm{sm}}(U, \mathbf{R})$. $\diamond$

By Lemma (5.16), the map F_σ is, in particular, Borel measurable. Hence $\operatorname{smear}_M(\sigma)$ is a well-defined measure on $\operatorname{map}_\infty(\Delta^k, M)$.

Lemma (5.18). *1. For each $k \in \mathbf{N}$ and each smooth chain $c \in C_k^{\mathrm{sm}}(U, \mathbf{R})$ we have* $\operatorname{smear}_M(c) \in \mathcal{C}_k(M)$.

2. The map $\operatorname{smear}_M \colon C_*^{\mathrm{sm}}(U, \mathbf{R}) \longrightarrow \mathcal{C}_*(M)$ *is a chain map.*

Proof. 1. Of course, we need only show this assertion for smooth singular simplices $\sigma \in \operatorname{map}_\infty(\Delta^k, U)$. According to Lemma (3.4), h_M is a signed measure of finite total variation because $h_M \geq 0$ and
$$h_M\big(\pi_1(M) \setminus \operatorname{Isom}_\infty^+(U)\big) = 1.$$
Furthermore, $\pi_1(M) \setminus \operatorname{Isom}_\infty^+(U)$ is a compact determination set of h_M. Now Lemma (3.9) yields
$$\operatorname{smear}_M(\sigma) = h_M^{F_\sigma} \in \mathcal{C}_k(M)$$
since F_σ is, due to Lemma (5.16), continuous.

2. Let $k \in \mathbf{N}$, $j \in \{0, \dots, k+1\}$, and let $\sigma \in \operatorname{map}_\infty(\Delta^{k+1}, U)$ be a smooth singular $(k+1)$-simplex. By linearity, it is sufficient to show
$$\operatorname{smear}_M(\sigma \circ \partial_j) = \partial_j\big(\operatorname{smear}_M(\sigma)\big)$$
where $\sigma \circ \partial_j$ stands for the j-th face of σ. By construction,
$$\begin{aligned}
\operatorname{smear}_M(\sigma \circ \partial_j) &= h_M^{F_{\sigma \circ \partial_j}} \\
&= h_M^{\big((\pi_1(M)) \cdot f \mapsto p_M \circ f \circ \sigma \circ \partial_j\big)} \\
&= \big(h_M^{F_\sigma}\big)^{(\tau \mapsto \tau \circ \partial_j)} \\
&= \partial_j\big(h_M^{F_\sigma}\big) \\
&= \partial_j\big(\operatorname{smear}_M(\sigma)\big). \qquad \square
\end{aligned}$$

To pull back the smearing operation to smooth singular simplices on N via p_N-lifts, we need the right invariance of the measure h_M to guarantee well-definedness:

Lemma (5.19). *Let $k \in \mathbf{N}$ and $\sigma \in \operatorname{map}_\infty(\Delta^k, N)$.*

1. *Then there is a p_N-lift of σ lying in $\operatorname{map}_\infty(\Delta^k, U)$. In fact, each continuous p_N-lift of σ is smooth.*
2. *If $\sigma_1, \sigma_2 \in \operatorname{map}_\infty(\Delta^k, U)$ are two p_N-lifts of σ, then*

$$h_M^{F_{\sigma_1}} = h_M^{F_{\sigma_2}}.$$

Proof. 1. Since Δ^k is simply connected, there is a continuous p_N-lift

$$\widetilde{\sigma}\colon \Delta^k \longrightarrow U$$

of σ. Noting that the covering projection $p_N\colon U \longrightarrow N$ is a local diffeomorphism (even a local isometry), we deduce from the commutative diagram

$$\begin{tikzcd} & U \arrow[d, "p_N"] \\ \Delta^k \arrow[ur, "\widetilde{\sigma}"] \arrow[r, "\sigma"'] & N \end{tikzcd}$$

that $\widetilde{\sigma}$ also must be smooth (smoothness is a local property).

2. Since Δ^k is connected, two continuous p_N-lifts of σ only differ by some element in $\pi_1(N)$. As already mentioned in Corollary (5.10), the fundamental group $\pi_1(N)$ acts by orientation preserving isometries on the universal cover U. Hence there is a $g \in \operatorname{Isom}^+_\infty(U)$ such that

$$\sigma_1 = g \circ \sigma_2.$$

Writing $r_g\colon \pi_1(M) \setminus \operatorname{Isom}^+_\infty(U) \longrightarrow \pi_1(M) \setminus \operatorname{Isom}^+_\infty(U)$ for right multiplication by g, we get $F_{\sigma_1} = F_{\sigma_2} \circ r_g$. Thus,

$$h_M^{F_{\sigma_1}} = h_M^{F_{\sigma_2} \circ r_g} = \big(h_M^{r_g}\big)^{F_{\sigma_2}}.$$

Finally, the right invariance of h_M enters the scene: Since h_M is right invariant, we have

$$h_M^{r_g} = h_M,$$

which completes the proof. □

Hence we can pull back smear_M to smooth singular chains on N:

Definition (5.20). For each $k \in \mathbf{N}$ and each smooth singular simplex $\sigma \in \operatorname{map}_\infty(\Delta^k, N)$ we define its **smeared chain** by

$$\operatorname{smear}_{N,M}(\sigma) := \operatorname{smear}_M(\widetilde{\sigma}) \in \mathcal{C}_k(M),$$

where $\widetilde{\sigma} \in \operatorname{map}_\infty(\Delta^k, U)$ is some p_N-lift of σ. Again, the notation $\operatorname{smear}_{N,M}$ will also be used for its linear extension $C^{\mathrm{sm}}_*(N, \mathbf{R}) \longrightarrow \mathcal{C}_*(M)$. ◇

Lemma (5.21). *The smearing map*

$$\operatorname{smear}_{N,M} \colon C^{\mathrm{sm}}_*(N, \mathbf{R}) \longrightarrow \mathcal{C}_*(M)$$

is a chain map.

Proof. Let $k \in \mathbf{N}$, $j \in \{0, \dots, k+1\}$, and let $\sigma \in \operatorname{map}_\infty(\Delta^{k+1}, N)$. If $\widetilde{\sigma}$ is a p_N-lift, then $\widetilde{\sigma} \circ \partial_j$ is obviously a p_N-lift of $\sigma \circ \partial_j$. Hence we conclude from Lemma (5.18)

$$\begin{aligned}
\operatorname{smear}_{N,M}(\sigma \circ \partial_j) &= \operatorname{smear}_M(\widetilde{\sigma} \circ \partial_j) \\
&= \partial_j\bigl(\operatorname{smear}_M(\widetilde{\sigma})\bigr) \\
&= \partial_j\bigl(\operatorname{smear}_{N,M}(\sigma)\bigr).
\end{aligned}$$

Thus $\operatorname{smear}_{N,M}$ is a chain map. □

Furthermore, smearing does not spoil the norm:

Lemma (5.22). *For all $k \in \mathbf{N}$ and all $c \in C^{\mathrm{sm}}_k(N, \mathbf{R})$*

$$\|\operatorname{smear}_{N,M}(c)\| \leq \|c\|^{\mathrm{sm}}_1 .$$

Proof. Since the measure h_M is positive, so is $\operatorname{smear}_{N,M}(\sigma)$ for all smooth singular simplices $\sigma \in \operatorname{map}_\infty\bigl(\Delta^k, N\bigr)$. This implies

$$\begin{aligned}
\|\operatorname{smear}_{N,M}(\sigma)\| &= \operatorname{smear}_{N,M}(\sigma)\bigl(\operatorname{map}_\infty(\Delta^k, U)\bigr) \\
&= h_M\bigl(\pi_1(M) \setminus \operatorname{Isom}^+_\infty(U)\bigr) \\
&= 1.
\end{aligned}$$

Now the general case is an immediate consequence of the triangle inequality for the total variation (compare Lemma (3.7)). □

The integration operation introduced in Section 4.4 makes it possible to understand the effect $\operatorname{smear}_{N,M}$ has on homology:

Theorem (5.23). *Smearing induces multiplication by* $\operatorname{vol}(N)/\operatorname{vol}(M)$ *on the top homology group. More precisely: for all cycles* $z \in C_n^{\mathrm{sm}}(N, \mathbf{R})$ *with* $[z] = [N]_{\mathrm{sm}}$ *in* $H_n^{\mathrm{sm}}(N, \mathbf{R})$,

$$\bigl[\operatorname{smear}_{N,M}(z)\bigr] = \frac{\operatorname{vol}(N)}{\operatorname{vol}(M)} \cdot [M]_{\mathrm{mh}} \in \mathcal{H}_n(M).$$

The proof basically relies on the following calculation:

Lemma (5.24). *For all* $\sigma \in \operatorname{map}_\infty(\Delta^k, N)$ *the following integrals are equal:*

$$\int_{\operatorname{smear}_{N,M}(\sigma)} \operatorname{vol}_M = \int_\sigma \operatorname{vol}_N .$$

Proof (of Lemma (5.24)). Let $\widetilde{\sigma}$ be some p_N-lift of σ. By definition, the integral can be computed as follows:

$$\begin{aligned}\int_{\operatorname{smear}_{N,M}(\sigma)} \operatorname{vol}_M &= \int_{\operatorname{map}_\infty(\Delta^k,M)} \int_\tau \operatorname{vol}_M \, d\bigl(\operatorname{smear}_{N,M}(\sigma)\bigr)(\tau) \\ &= \int_{\operatorname{map}_\infty(\Delta^k,M)} \int_\tau \operatorname{vol}_M \, d\bigl(h_M^{F_{\widetilde{\sigma}}}\bigr)(\tau).\end{aligned}$$

Using the transformation formula for measures, we get

$$\begin{aligned}\int_{\operatorname{smear}_{N,M}(\sigma)} \operatorname{vol}_M &= \int_{\pi_1(M)\backslash \operatorname{Isom}_\infty^+(U)} \int_{F_{\widetilde{\sigma}}(\pi_1(M)\cdot f)} \operatorname{vol}_M \, dh_M(\pi_1(M)\cdot f) \\ &= \int_{\pi_1(M)\backslash \operatorname{Isom}_\infty^+(U)} \int_{p_M \circ f \circ \widetilde{\sigma}} \operatorname{vol}_M \, dh_M(\pi_1(M)\cdot f) \\ &= \int_{\pi_1(M)\backslash \operatorname{Isom}_\infty^+(U)} \int_{\widetilde{\sigma}} f^* p_M^* \operatorname{vol}_M \, dh_M(\pi_1(M)\cdot f).\end{aligned}$$

By definition, p_M and p_N are orientation preserving local isometries. Since orientation preserving local isometries between Riemannian manifolds pull back the volume form to the volume form, we conclude

$$\begin{aligned}
\int_{\mathrm{smear}_{N,M}(\sigma)} \mathrm{vol}_M &= \int_{\pi_1(M)\backslash \mathrm{Isom}^+_\infty(U)} \int_{\widetilde{\sigma}} \mathrm{vol}_U \; dh_M(\pi_1(M)\cdot f) \\
&= \int_{\pi_1(M)\backslash \mathrm{Isom}^+_\infty(U)} \int_{\widetilde{\sigma}} p_N^* \,\mathrm{vol}_N \; dh_M \\
&= \int_{\pi_1(M)\backslash \mathrm{Isom}^+_\infty(U)} dh_M \cdot \int_{\widetilde{\sigma}} p_N^* \,\mathrm{vol}_N \\
&= 1 \cdot \int_{p_N\circ\widetilde{\sigma}} \mathrm{vol}_N \\
&= \int_\sigma \mathrm{vol}_N\,. \qquad \square
\end{aligned}$$

Proof (of Theorem (5.23)). Lemma (5.24) and Lemma (1.10) imply

$$\int_{\mathrm{smear}_{N,M}(z)} \mathrm{vol}_M = \int_z \mathrm{vol}_N = \mathrm{vol}(N).$$

Application of Theorem (4.35) therefore results in

$$\big[\mathrm{smear}_{N,M}(z)\big] = \frac{\mathrm{vol}(N)}{\mathrm{vol}(M)} \cdot [M]_{\mathrm{mh}} \in \mathcal{H}_n\,(M). \qquad \square$$

Remark (5.25). It is possible to extend the smearing chain map $\mathrm{smear}_{N,M}$ to all measure chains:

$$\begin{aligned}
\mathcal{C}_k\,(N) &\longrightarrow \mathcal{C}_k\,(M) \\
\mu &\longmapsto \Big(A \mapsto \int_{\mathrm{map}_\infty(\Delta^k,N)} h_M^{F_{\widetilde{\sigma}}}(A)\; d\mu(\sigma)\Big).
\end{aligned}$$

This smearing map also does not increase the norm and induces multiplication by $\mathrm{vol}(N)/\,\mathrm{vol}(M)$ on top homology. This leads to an intrinsic proof for the proportionality principle of the "simplicial volume" defined via measure homology. $\diamond$

5.5 ■ Proof of the proportionality principle

Combining the techniques from Section 5.4 and the isometric isomorphism of Section 4.3, we are now in state to do the final step in the proof of the proportionality principle of simplicial volume:

Proof (of Theorem (5.1)). Suppose $z \in C_n^{\mathrm{sm}}(N, \mathbf{R})$ is a cycle which represents the (smooth) fundamental class $[N]_{\mathrm{sm}}$. Theorem (5.23) shows that $\mathrm{smear}_{N,M}$ induces the factor $\mathrm{vol}(N)/\mathrm{vol}(M)$ on top homology, i.e.,

$$\big[\mathrm{smear}_{N,M}(z)\big] = \frac{\mathrm{vol}(N)}{\mathrm{vol}(M)} \cdot [M]_{\mathrm{mh}}\,.$$

Since we can compute the simplicial volume via measure homology (Corollary (4.15)) and since smearing does not increase the norm (Lemma (5.22)), we obtain

$$\begin{aligned}
\|M\| &= \|[M]_{\mathrm{mh}}\|_{\mathrm{mh}} \\
&\leq \frac{\mathrm{vol}(M)}{\mathrm{vol}(N)} \cdot \|\mathrm{smear}_{N,M}(z)\| \\
&\leq \frac{\mathrm{vol}(M)}{\mathrm{vol}(N)} \cdot \|z\|_1^{\mathrm{sm}}\,.
\end{aligned}$$

Now taking the infimum over all (smooth) representatives of $[N]_{\mathrm{sm}}$ gives the estimate

$$\|M\| \leq \frac{\mathrm{vol}(M)}{\mathrm{vol}(N)} \cdot \|[N]_{\mathrm{sm}}\|_1^{\mathrm{sm}} = \frac{\mathrm{vol}(M)}{\mathrm{vol}(N)} \cdot \|N\|$$

because $\|[N]_{\mathrm{sm}}\|_1^{\mathrm{sm}} = \|N\|$ by Corollary (4.4). Swapping the rôles of the manifolds M and N shows the reverse inequality

$$\|N\| \leq \frac{\mathrm{vol}(N)}{\mathrm{vol}(M)} \cdot \|M\|$$

and hence proves the proportionality principle. □

5.6 ■ Applications of the proportionality principle

Using Gromov's upper bound for the simplicial volume in terms of the minimal volume (Theorem (1.29)), we already have seen that the simplicial volume of flat oriented closed connected manifolds must vanish. We will give now a proof of this fact based on the proportionality principle instead of applying Gromov's sophisticated estimate:

Corollary (5.26). *If M is an oriented closed connected flat Riemannian manifold (i.e., it is locally isometric to Euclidean space), then*

$$\|M\| = 0.$$

Proof. For $c \in \mathbf{R}_{>0}$ we consider the Riemannian manifold M_c which is obtained from M by scaling its Riemannian metric by the factor c^2 (i.e., all lengths are stretched by the factor c). Then M and M_c are homeomorphic manifolds (in particular, $\|M\| = \|M_c\|$) and

$$\operatorname{vol}(M_c) = c^n \cdot \operatorname{vol}(M)$$

where n is the dimension of M. Moreover, M_c is also flat. Therefore both universal covers $\widetilde{M}$ and $\widetilde{M_c}$ are isometric to $\mathbf{R}^n$ [Lee$_1$, Theorem 11.12]. Thus the proportionality principle yields

$$\frac{\|M\|}{\operatorname{vol}(M)} = \frac{\|M_c\|}{\operatorname{vol}(M_c)} = \frac{\|M\|}{c^n \cdot \operatorname{vol}(M)}.$$

Since this holds for all $c \in \mathbf{R}_{>0}$, clearly $\|M\| = 0$ follows. □

Since the simplicial volume is a homotopy invariant, we can deduce from the proportionality principle that the Riemannian volume in some special cases is homotopy invariant:

Corollary (5.27). *If M and N are homotopy equivalent oriented closed connected Riemannian manifolds with isometrically isomorphic universal Riemannian coverings and $\|M\| \neq 0$, then*

$$\operatorname{vol}(M) = \operatorname{vol}(N).$$ □

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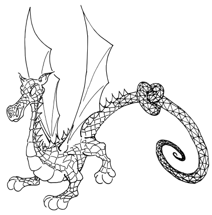


Table of Notation

Symbols

T

$T\Delta^k$	$= \Delta^k \times \mathbf{R}^k$	57
Tf	differential of f	

U

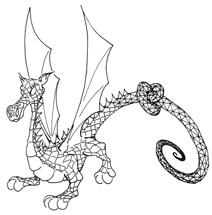
u^*		45
U^K		54

V

V_ε		91
$\widetilde{V}_\varepsilon$		91
V^G	G-fixed points in V	37
v_n	maximal volume of an ideal n-simplex in hyperbolic space	12
$\mathrm{vol}(M)$	volume of M	
vol_M	volume form of M	

Z

$\mathbf{Z}$	integers	



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